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A TAUBERIAN THEOREM AND ITS APPLICATION TO CONVERGENCE OF FOURIER SERIES

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1. Let $\left(1+\frac{x}{2}+\frac{x^2}{3}+\cdots\frac{x^n}{n+1}+\text{ to }\infty\right)^k=\sum_0^\infty b_{k,n}\,x^n$, k being a positive integer and let $\sum_{r=0}^{n} b_{k,r}=B_{k,n}$. Whenever $\sum_0^n b_{k,n-r}\cdot S_r/B_{k,n}$ converges to a limit say S as $n\to\infty$, I shall indicate that fact by the symbol $S_n\to S\ (N, k)$ as $n\to\infty$. (1·1)

The main object of this note is the proof of two theorems (I and II) the first of these being a Tauberian theorem on mean convergence or summability (N, k), the second being the summability (N, 1) of the Fourier series of a summable (L) function $f(x)$ at a point x_0 at which, $\{\left(f(x_0+h)-f(x_0)\right)\cdot\log|h|\}\to 0$ as $h\to 0$. From these two theorems, which are new, it is noteworthy that the truth of the convergence criterion, of Hardy and Littlewood (1932) of the Fourier series of a summable function $f(x)$ at a point x_0, namely (1) $\{\left(f(x_0+h)-f(x_0)\right)\log|h|\}\to 0$ as $h\to 0$ (2) the Fourier coefficients of $f(x)$ or order $n\leqslant A\cdot n^{-\delta}$, $(A>0,\ \delta>0)$, (1·2) follows at once. Before proceeding further it should be noted that summability (N, k) is more stringent than Cesaro means of any positive order, for it can be established that (N, k) implies (C, r) when $k>0$ and $r>0$, but not conversely.

2. *Statement of Theorems*

Theorem I. If $S_n\to S\ (N, k)$ k being a positive integer and $S_n-S_{n-1}=\delta S_n\leqslant An^{-\mu}$ $(A>0,\ 1>\mu>0)$, then S_n converges ordinarily to S. (2·1)

Note 1.—If $\mu=1$, in view of the remark at the end of 1, the theorem reduces itself to Cesaro-Tauber theorem.

Note 2.—As we should expect from Hardy-Littlewood's paper,[1] we should be able to prove that whenever $S_n\to S\ (N, k)$, k being a positive integer then $S_n\to S$ also by Valeron means (V, k_1) of order k_1 where

$$1\leqslant k_1<2, \textit{ i.e.,}\ \frac{1}{\sqrt{\pi}\cdot M}\sum_{p=0}^{\infty} S_p\, e^{\frac{-(p-n)^2}{M}}\to S \text{ as } n\to\infty,\ M=2\cdot n^{2-k_1};\quad (2\cdot 2)$$

then the Tauberian Theorem I follows. But the direct proof of Theorem I, which is given here, apart from its intrinsic interest is much simpler than the proof of the fact that (N, k) implies (V, k_1) for k a positive integer and $1 \leqslant k_1 < 2$. However for completeness the fact of (N, k) implying (V, k_1) is stated as Theorem III, and the proof of this theorem under the title "On the relation between summability by Norlund means of a certain type and summability by Valeron Means," will appear shortly in the *Journal of the Mysore University*.

THEOREM II. If $f(x)$ be a periodic (2π) L integeble function and $\{\big(f(x_0+h)-f(x_0)\big) \log |h|\} \to 0$ as $h \to 0$, then the Fourier series of $f(x)$ at x_0 is summable (N, 1) to $f(x_0)$. (2·3)

THEOREM III. If $S_n \to S$, (N, k) k being a positive integer then

$$S_n \to S, \ (V, k_1) \qquad 1 \leqslant k_1 < 2 \text{ (for all such } k_1). \tag{2·4}$$

3. Proof of Preliminary Lemmas

Lemma I. Let $b_{k,n}$ and $B_{k,n}$ be as in (1·1), and $a_{k,n}$ be defined by

$$\Big(1 - \sum_{r=1}^{\infty} a_{k,r}\, x^r\Big)\Big(\sum_{0}^{\infty} b_{k,r}\, x^r\Big) = 1 \tag{3·1}$$

then
$$\left.\begin{aligned} &(a)\ b_{k,n} = \sum_{r=1}^{n} a_{k,r} \cdot b_{k,n-r}, \qquad (b)\ B_{k,n} = 1 + \sum_{r=1}^{n} a_{k,r}\, B_{k,n-r} \\ &(c)\ b_{k,n} = O\Big(\frac{(\log n)^{k-1}}{n}\Big) \qquad (d)\ a_{k,n} = O\Big(\frac{1}{n \cdot (\log n)^{1+k}}\Big). \end{aligned}\right\} \tag{3·2}$$

(*a*) and (*b*) follow from the definition in (3·1) and (*c*) can be deduced by induction from

$$b_{k,n}\, x^n = \Big(1 + \sum_{1}^{\infty} \frac{x^r}{r+1}\Big)^k. \tag{3·3}$$

Proof of (*d*) *of* (3·2)

Let $V_n(\alpha) = \dfrac{(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{\underline{|n}}$; we shall here establish by induction that

$$a_{k,n} = \frac{1}{\underline{|k-1}} \int_0^1 V_{n-k} \cdot \alpha^k \cdot d\alpha + O\Big(\frac{1}{n^2 \cdot (\log n)^2}\Big). \tag{3·4}$$

$[k \geqslant 1]$

Let $I = \displaystyle\int_0^1 (1-x)^{\alpha} \cdot \frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\, d\alpha$; integrating directly and expanding

$(1-x)^{\alpha}$ and integrating term by term, we obtain

$$I=\frac{(1-x)-\sum_{r=0}^{k-1}\frac{(\log 1-x)^r}{\underline{|r}}}{(\log 1-x)^k}$$

$$=\frac{1}{\underline{|k}}-\sum_{n=1}^{\infty}\left[\int_0^1 V_n(\alpha)\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\,d\alpha\right\}\cdot x^n \quad (3\cdot5)$$

Multiplying by $(-x)^{k-1}$, we obtain from 3·5

$$\left(\frac{-x}{\log 1-x}\right)^k=\sum_{r=1}^{k-1}\frac{(-x)^{k-1-r}}{\underline{|k-r}}\cdot\left(\frac{-x}{\log 1-x}\right)^r$$

$$+(-1)^{k-1}\left\{\frac{x^{k-1}}{\underline{|k}}-\sum_{n=1}^{\infty}\left\{V_n(\alpha)\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\cdot d\alpha\right\}x^{n+k-1}\right\} \quad (3\cdot6)$$

From 3·1 and 1·1 definitions we have

$$\left(\frac{-x}{\log 1-x}\right)^k=1-\sum_{n=1}^{\infty}a_{k,n}\,x^n \quad (3\cdot7)$$

so that for $n>k$ we obtain from 3·6

$$a_{k,n}=a_{k-1,n}-\frac{a_{k-2,n-1}}{\underline{|2}}+\cdots+\cdots$$

$$+(-1)^{k-2}\frac{a_{1,n-k+2}}{\underline{|k-1}}+(-1)^{k-1}\cdot\int_0^1 V_{n-k+1}^{(\alpha)}\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\,d\alpha. \quad (3\cdot8)$$

It is easy to see that (3·4) is true for $k=1$; assume it true for $k=1, 2,\cdots k-1$, then from 3·8

$$a_{k,n}=\int_0^1 V_{n-k+1}\left(\frac{\alpha^{k-1}}{\underline{|k-2}}-\frac{\alpha^{k-2}}{\underline{|2}\ \underline{|k-3}}+\cdots\frac{(-1)^{k-2}}{\underline{|k-1}}\cdot\alpha\right.$$

$$\left.+(-1)^{k-1}\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\cdot\right)d\alpha+O\left(\frac{1}{n^2\ (\log n)^2}\right) \quad (3\cdot9)$$

$$=\int_0^1 V_{n-k+1}^{(\alpha)}\cdot\alpha^k\,d\alpha+O\left(\frac{1}{n^2\cdot(\log n)^2}\right). \quad (3\cdot10)$$

Now $V_n\leqslant\frac{K}{n^{1+\alpha}}$ and $V_{n-k+1}=V_{n-k}\left\{1+O\left(\frac{1}{n}\right)\right\}$.

Hence $$\int_0^1 V_{n-k+1}\,\alpha^k\,d\alpha=\int_0^1 V_{n-k}\cdot\alpha^k\,d\alpha+O\left\{\frac{1}{n}\cdot\int_0^1 V_n\cdot\alpha^k\,d\alpha.\right\}$$

$$=\int_0^1 V_{n-k}\cdot\alpha^k\cdot d\alpha+O\left(\frac{1}{n^2\cdot(\log n)^{1+k}}\right) \quad (3\cdot11)$$

Hence $$a_{k,n} = \int_0^1 V_{n-k}\, a^k \cdot da + O\left(\frac{1}{n^2 \cdot (\log n)^2}\right) \text{ for } k \geqslant 1 \qquad (3\cdot12)$$

thus proving (3·4)

From (3·12)

$$a_{k,n} = O\left(\frac{1}{n \cdot (\log n)^{1+k}}\right) + O\left(\frac{1}{n^2 \cdot (\log n)^2}\right)$$

$$= O\left(\frac{1}{n \cdot (\log n)^{1+k}}\right) \text{ thus proving } (d) \text{ of } (3\cdot2) \qquad (3\cdot13)$$

4. Lemma II

If $S_n \to 0$ (N, k) where k is a positive integer and $N = N_n = [K \cdot n^{\delta}]$ where $0 > \delta > 1$, and K is a positive fixed number, then

$$\left[\sum_{r=0}^{n} b_{k,\, r+N} \cdot S_{n-r} \Big/ \sum_{0}^{n} b_{k,\, r+N}\right] \to 0 \text{ as } n \to \infty \qquad (4\cdot1)$$

[Since in what follows k is a fixed integer, a_n, b_n, and B_n, shall stand for $a_{k,n}$, $b_{k,n}$ and $B_{k,n}$ respectively and whenever x_n and y_n are of the same order we shall indicate by $x_n = O(S_n)$ (big O) and whenever $\frac{x_n}{y_n} \to 0$, we indicate by $x_n = 0(y_n)$ (little 0).] (3·2)

From 3·2 and the condition about N it is easy to deduce that

$$\sum_{r=0}^{n} b_{k,\, r+N} = \sum_{0}^{n} b_{r+N} \text{ (by, 4·2)} = O(B_{k,n}) = O[(\log n)^k] \qquad (4\cdot3)$$

and we have therefore to prove $\sum_{r=0}^{n} b_{r+N}\ S_{n-r} = 0\ (B_n)$. (4·4)

Let $\sum_{0}^{n} b_r\ S_{n-r}/B_n = y_n$; then from relation (3·1) we obtain

$$S_n = B_n\, y_n - \sum_{r=1}^{n} a_r \cdot B_{n-r}\, y_{n-r}. \qquad (4\cdot5)$$

Substituting (4·5) in the l.h.s..of 4·4 we obtain

$$\sum_{r=0}^{n} b_{r+N}\, S_{n-r} = \sum_{r=0}^{n} A_{n,r} \cdot B_r \cdot y_r \qquad (4\cdot6)$$

where $$A_{n,r} = b_{n-r+N} - a_1\, b_{n-r-1+N} - a_{n-r}\, b_N. \qquad (4\cdot7)$$

We shall now consider r.h.s of 4·6 in three parts.

$$\sum_{r=0}^{n} A_{n,r} \cdot B_r \cdot y_r = \sum_{r=0}^{r=n_1} + \sum_{n_1+1}^{[n-N^{1-\epsilon}]} + \sum_{[n-N^{1-\epsilon}]+1}^{n} = A + B + C. \qquad (4\cdot8)$$

where n_1 is a conveniently chosen fixed integer.

Since by (3·2) $\Sigma\,|\alpha_r|$ converges, we have from (4·7) and (3·2)

$$|\,A_{n,r}\,| \leqslant O\left(\frac{(\log N)^{k-1}}{N}\right). \tag{4·9}$$

Hence $$|\,C\,| \leqslant K \cdot \sum_{0 \leqslant n-r \leqslant N^{1-\epsilon}} A_{n,r}\,B_r \cdot \leqslant K' \cdot N^{1-\epsilon} \cdot \frac{(\log N)^{k-1}}{N} \cdot B_n = 0\,(B_n). \tag{4·10}$$

Consider now B. From 4·7 and 3·2 (*a*) we have

$$A_{n,r} = a_{n-r+1}\,b_{N-1} + \cdots + a_{n-r+N} \tag{4·11}$$

Since in B

$$n - r \geqslant N^{1-\epsilon} - 1 \text{ and } a_n = O\left(\frac{1}{n \cdot (\log n)^{1+k}}\right) \text{ and } b_n = O\left(\frac{(\log n)^{k-1}}{n}\right)$$

and $N = [K \cdot n^{\delta}]$ $(0 < \delta < 1)$ from, (3·2)

we obtain from (4·11) $|\,A_{n,r}\,| \leqslant K_1 \sum_{p=1}^{p=N} \dfrac{(\log N - p + 2)^{k-1}}{(N-p+2)\,(q+p)\,[\log (q+p)]^{1+k}}$

where $q = n - r$.

$$\leqslant \frac{K_2}{(\log n)^2} \cdot \sum_{p=1}^{p=N} \frac{1}{(q+p)\,(N-p+2)} \tag{4·12}$$

$$\leqslant \frac{K_3}{\log n \cdot (N+q+2)} \tag{4·13}$$

so that $$|\,B\,| \leqslant \underset{(n_1+1 \leqslant r \leqslant n-N^{1-\epsilon})}{\text{Max of } y_r} \cdot \frac{K_3 \cdot B_n}{\log n} \cdot \sum_{q=n}^{q=\infty} \frac{1}{N+2+q}$$

$$\leqslant \underset{n_1+1 \leqslant r}{\text{Max of } y_r} \cdot K_4 \cdot B_n. \tag{4·14}$$

Consider now A. From (4·13) for fixed r as $n \to \infty$ we have

$$|\,A_{n,r}\,| \leqslant \frac{K_4}{n \cdot \log n} \tag{4·15}$$

so that for fixed r $$A_{n,r} \cdot B_r = 0\,(B_n) \tag{4·16}$$

so that from (4·10), (4·14) and (4·16) and hypothesis that $y_n \to 0$ as $n \to \infty$ (*i.e.*, 4·1) we obtain $A + B + C = 0\,(B_n)$, thus proving 4·4 and establishing Lemma II. (4·17)

5. Proof of Theorem I

Without loss of generality we assume that

$$\sum_{0}^{n} b_{n-r}\,S_r/B_n \to 0, \text{ as } n \to \infty. \tag{5·1}$$

[As in 4, a_n, b_n, B_n stand for $a_{k,n}$, $b_{k,n}$ and $B_{k,n}$].

Suppose lim inf $S_n = -b$, $b > 0$ (5·2)

Then there exists a sequence $n_0<n_1<\cdots<n_r\to\infty$ such that $S_{n_r}\leqslant -b_1$ $0<b_1<b$. (5·3)

Since $\delta S_n\leqslant A\cdot n^{\mu}$ $(0<\mu<1)$ it can be easily shown that

$$S_n\leqslant -b_1/2 \text{ when } n_r\leqslant n\leqslant n_r+N_{n_r} \tag{5·4}$$

$$\text{where } N_n=N=\left[\frac{b_1}{2}\cdot\frac{n^{\mu}}{A}\right]$$

Now $$\sum_{r=0}^{n+N} b_{n+N-r}\,S_r=\sum_{r=0}^{r=n}+\sum_{r=n+1}^{n+N}=\Sigma_1+\Sigma_2 \tag{5·5}$$

From Lemma II $$\Sigma_1=0\ (B_{n+N}). \tag{5·6}$$

By taking $n=n_r$, in view of 5·4

we have $$\Sigma_2\leqslant -\frac{b_1}{2}\cdot\sum_{r=n+1}^{r=N+n} b_{n+N-r}=-\frac{b_1}{2}\cdot\sum_{p=0}^{N-1} b_p=-\frac{b_1}{2}\cdot B_{N-1}. \tag{5·7}$$

In view of condition about N, it can be shewn that $B_{N-1}\geqslant K_1\cdot B_{n+N}$ (5·8)

where K_1 is a positive constant.

Hence $\Sigma_2/B_{n_r+N_{n_r}}\leqslant -\frac{b_1}{2}\cdot K_1\cdot$ so that

$$\lim\ \inf\ \frac{\sum_0^n b_{n-r}\,S_r}{B_n}\leqslant -\frac{b_1}{2}\,K_1 \text{ contradicting } 5·1. \tag{5·9}$$

Hence *lim inf* $S_n=0$, the case of $\lim\inf S_n=-\infty$ being easily disposed of. (5·10)

similarly $\lim\sup S_n=0$, thus establishing Theorem I. (5·11)

6. Proof of Theorem II

Let $S_n=\sum_0^n (a_r\cos rx+b_r\sin rx)$, a_r, b_r, being the Fourier coefficients of $f(x)$, and $$\psi(t)=\frac{f(x_0+t)+f(x_0-t)-2f(x_0).}{2} \tag{6·1}$$

Then $$S_n(x_0)-f(x_0)=\frac{1}{\pi}\int_0^{\pi}\psi(t)\cdot\frac{\sin(n+\frac{1}{2})t}{\sin t/2}\cdot dt=\frac{1}{\pi}\int_0^{\delta}+\epsilon(n,\delta)$$

where δ is so chosen that $|\psi(t)\log t|\leqslant\epsilon$ $0\leqslant t\leqslant\delta$; $\epsilon(n,\delta)\to 0$ as $n\to\infty$ (6·2)

Now $$\sum_{r=0}^{n}\frac{S_{n-r}(x_0)}{r+1}\Big/\sum_{r=0}^{n}\frac{1}{r+1}-f(x_0)=\frac{1}{\pi L_n}\int_0^{\delta}\psi(t)\cdot\frac{K_n(t)}{\sin t/2}\,dt$$

$$+\epsilon'(n,\delta) \tag{6·3}$$

where $$K_n(t)=\sum_{r=0}^{n}\frac{\sin(r+\frac{1}{2})t}{n+1-r}, \text{ and } L_n=\sum_{r=0}^{n}\frac{1}{1+r} \tag{6·4}$$

We shall first prove that

$$\left.\begin{aligned} &\left|\frac{K_n(t)}{\sin t/2}\right| \leqslant L_n \cdot 2n+1 \text{ in } 0 \leqslant t \leqslant \frac{\pi}{n} \\ \text{and} \quad &|K_n(t)| \leqslant K\,|\log t| \text{ in } \frac{\pi}{n} \leqslant t \leqslant \delta < 1 \end{aligned}\right\} \quad (6\cdot5)$$

The first part follows from the fact that $\left|\frac{\sin pt}{\sin t}\right| \leqslant p$ when p is an integer.

To prove the second part we note that

$$K_n(t) = \text{Imaginary part of}\left\{-e^{-i(n+3/2)t}\cdot\sum_{p=1}^{n+1}\frac{e^{ipt}}{p}\right\} =$$

$$= \text{Imaginary part of}\left\{+e^{-i(n+3/2)t}\cdot\left(\log(1-e^{it}) + \sum_{p=n+2}^{\infty}\frac{e^{ipt}}{p}\right)\right\}$$

$$= \text{Imaginary part of}\left\{e^{-i(n+3/2)t}(P_1+P_2)\right\} \quad (6\cdot6)$$

Now $|P_1| = O\{|\log t|\}$ when $0 \leqslant t \leqslant \delta < 1$

and in $\frac{\pi}{n} \leqslant t \leqslant \delta < 1$ $|P_2| \leqslant \frac{K}{n\cdot t} = O(1)$, so that second part of 6·5 is established. (6·7)

Now $$\int_0^{\delta}\psi(t)\frac{K_n(t)}{\sin t/2}\,dt = \int_0^{\pi/n} + \int_{\pi/n}^{\delta}$$

$$\left.\begin{aligned} &(6\cdot2) \text{ and } (6\cdot5) \text{ imply that } \left|\int_0^{\pi/n}\right| \leqslant K\cdot\epsilon\cdot L_n \text{ and} \\ &\left|\int_{\pi/n}^{\delta}\right| \leqslant K_1\int_{\pi/n}^{\delta}\frac{|\psi(t)\log t|}{t}\,dt \leqslant K_2\cdot\epsilon\cdot\log n \end{aligned}\right\} \quad (6\cdot8)$$

(6·3) and (6·8) implies that $S_n(x_0) \to f(x_0)$, (N, 1) thus establishing Theorem II.

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ON ARITHMETIC FUNCTIONS

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Introduction

In a paper on the theory of multiplicative arithmetic functions R. Vaidyanathaswamy has investigated two operations on arithmetic functions—'Composition' and 'Compounding'. In the same paper he has introduced the notion of a 'principal function' and has applied it to prove an interesting theorem on multiplicative functions which are functions of the g.c.d. of the arguments, the proof being by the method of 'generating series'. The objects of the present paper are (1) to study an operation which generalizes composition and compounding, (2) to study in detail principal functions, to give a purely arithmetic proof of Vaidyanathaswamy's result mentioned above and to obtain other similar results, (3) to evaluate certain multiple Dirichlet series, and (4) to evaluate certain multiple Lambert series.

The notions and notations in Vaidyanathaswamy's paper are employed here.

Section 1. *A Generalised Composition.*

1. Let (D) denote the matrix

$$\begin{pmatrix} d_{11} \cdots d_{1r} \\ \vdots \quad \vdots \\ d_{s1} \cdots d_{sr} \end{pmatrix}$$

and let f and ψ be two arithmetic functions of a $(r \times s)$ matrix set of arguments. Consider the function

$$\Sigma f(\mathrm{D}),$$

the summation being for all divisions d_{ij} such that $\prod_{i=1}^{s} d_{ij} = \mathrm{M}_j \ (j = 1, 2, \ldots r)$ and $\psi(\mathrm{D}) = 1$. We shall show that this operation on f includes composition and compounding as particular cases.

Let the number of rows in the matrix be two and suppose that

$$f\begin{pmatrix} \mathrm{M}_1, \ldots, \mathrm{M}_r \\ \mathrm{N}_1, \ldots, \mathrm{N}_r \end{pmatrix} = f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r) \times f_2(\mathrm{N}_1, \ldots, \mathrm{N}_r).$$

If we take ψ to be E (the function whose value is unity for all values of the arguments) it is clear that the above operation reduces to the composition of f_1 and f_2 $(f_1 \cdot f_2)$.

Let $g(M, N)$ denote the greatest common divisor (g.c.d.) of M and N. If we take $\psi\begin{pmatrix} M_1, & \ldots, & M_r \\ N_1, & \ldots, & N_r \end{pmatrix}$ to be $\prod_{i=1}^{r} g(M_i, N_i)$, then it can easily be seen that the generalized operation is simply the compounding of f_1 and f_2 $(f_1 \oplus f_2)$.

THEOREM 1: If f and ψ are multiplicative functions the latter being positive integral valued, then

$$\sum_{\substack{\psi(D)=1 \\ \prod_{i=1}^{r} d_{ig} = M_g \; (g=1,\cdots,r)}} f(D) = F(M_1, \ldots, M_r)$$

is a multiplicative function of r arguments.

Proof.—Let $\prod_{i=1}^{r} M_i$ be prime to $\prod_{i=1}^{r} N_i$ and let (G), (DG) denote the matrices

$$\begin{pmatrix} \delta_{11}, \ldots\ldots\ldots, & \delta_{1r} \\ \vdots & \vdots \\ \delta_{s1}, \ldots\ldots\ldots, & \delta_{sr} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_{11}\,\delta_{11}, \ldots\ldots\ldots, & d_{1r}\,\delta_{1r} \\ \vdots & \vdots \\ d_{s1}\,\delta_{s1}, \ldots\ldots\ldots, & d_{sr}\,\delta_{sr} \end{pmatrix}$$

respectively. Then it follows that

$$\begin{aligned} F(M_1N_1, \ldots, M_rN_r) &= \sum_{\substack{\psi(DG)=1 \\ \prod_{i=1}^{r} d_{ig}=M_g,\ \prod_{i=1}^{r} \delta_{ig}=N_g,\ (g=1,\cdots,r)}} f(DG) \\ &= \sum_{\psi(D)\,\psi(G)=1} f(D) f(G) \\ &= \{\sum_{\substack{\psi(D)=1 \\ \prod_{i=1}^{r} d_{ig}=M_g}} f(D)\} \times \{\sum_{\substack{\psi(G)=1 \\ \prod_{i=1}^{r} \delta_{ig}=N_g}} f(G)\} \quad (g=1,2,\cdots,r) \\ &= F(M_1, \ldots, M_r) \times F(N_1, \ldots, N_r) \end{aligned}$$

so that F is multiplicative.

2. The following are some other particular cases of the generalized composition.

(1) Let $g(M_1, \ldots, M_r)$ denote the g.c.d. of $M_1, \ldots, M_r$, and define $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ to be $g(M_1, \ldots, M_r) \times E(N_1, \ldots, N_r)$. Then ψ is clearly multiplicative. Therefore

$$\sum_{g(d_1, \cdots, d_r)=1\,;\; d_i \mid M_i} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1, \ldots, M_r/d_r \end{pmatrix}$$

is multiplicative if f is multiplicative.

(2) Take ψ to be $\prod_{i \neq g} g(M_i, M_g)\ E(N_1, \ldots, N_r)$. Then the multiplicativity of F follows that of f.

(3) Take $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ to be

$$\prod_{i=1}' g(M_i, m_i)\, E(N_1, \ldots N_r)$$

where m_i $(i = 1, \ldots, r)$ are given numbers. Then ψ is multiplicative. Therefore if f is multiplicative so also is the function

$$\sum_{\substack{g(d_i, m_i) = 1 \\ (d_i \mid M_i\,;\; i = 1, \cdots, r)}} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1 \ldots, M_r/d_r \end{pmatrix}.$$

(4) Let $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ be $g(M_1, \ldots, M_r, N_1, \ldots, N_r)$; then we see that if f be multiplicative

$$\sum_{g(d_1, \cdots, d_r, M_1/d_1, \cdots, M_r/d_r)=1,\; d_i \mid M_i} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1, \ldots, M_r/d_r \end{pmatrix}$$

is also multiplicative.

3. *Principal functions.*—f is called a principal function of r arguments equivalent to θ if

$$f(M_1, \ldots, M_r) = 0, \text{ unless } M_1 = \ldots = M_r,$$

and $f(M, \ldots, M) = \theta(M)$.

We shall write $f = \text{princ } \theta$.

THEOREM 2: If f_1, f_2 be two principal functions of r arguments equivalent to θ_1 and θ_2 respectively, then

$$f_1 \cdot f_2 = \text{princ } \theta_1 \cdot \text{princ } \theta_2 = \text{princ } (\theta_1 \cdot \theta_2),$$

and

$$f_1 \oplus f_2 = \text{princ } \theta_1 \oplus \text{princ } \theta_2 = \text{princ } (\theta_1 \oplus \theta_2).$$

All terms in the sum

$$\Sigma f_1(d_1, \ldots, d_r) f_2\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

where the summation is for any set of divisors d_i of M_i $(i = 1, \ldots, r)$ will vanish unless

$$d_1 = d_2 \ldots = d_r, \quad \frac{M_1}{d_1} = \ldots = \frac{M_r}{d_r}$$

simultaneously, and so it is a principal function. In particular it follows that the composite or the compound of two principal functions is a principal function.

Also

$$(\text{princ } \theta_1 \cdot \text{princ } \theta_2)(M, \ldots, M) = \sum_{d|M} f_1(d_1 \ldots, d) f_2\left(\frac{M}{d}, \ldots, \frac{M}{d}\right)$$

$$= \sum_{d|M} \theta_1(d)\, \theta_2\left(\frac{M}{d}\right) = (\theta_1 \cdot \theta_2)(M);$$

and

$$(\text{princ } \theta_1 \oplus \text{princ } \theta_2)(M, \ldots, M) = \sum_{d|M\,;\, g(d,\, M/d)=1} f_1(d, \ldots, d) f_2\left(\frac{M}{d}, \ldots, \frac{M}{d}\right)$$

$$= \sum_{d|M\,;\, g(d,\, M/d)=1} \theta_1(d)\, \theta_2(M/d) = (\theta_1 \oplus \theta_2)(M);$$

whence the required results follow.

THEOREM 3: The function

$$\Sigma f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right) \theta(d)$$

where the summation is for all common divisors d of $M_1, \ldots, M_r$ is equal to

$$(f \cdot \text{princ } \theta)(M_1, \ldots, M_r).$$

For $(f \cdot \text{princ } \theta)(M_1, \ldots, M_r) = \sum_{d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)(\text{princ } \theta)(d_1, \ldots, d_r)$

But $(\text{princ } \theta)(d_1, \ldots, d_r) = 0$ unless $d_1 = \ldots = d_r$,

$$= \theta(d) \text{ if } d_1 = \ldots = d_r = d,$$

so that $\sum_{d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)(\text{princ } \theta)(d_1, \ldots, d_r)$

$$= \Sigma f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right)\theta(d)$$

the summation being for all common divisors d of $M_1, \ldots, M_r$.

In a similar manner we prove

THEOREM 4: The function

$$\sum_{d|M_1, \cdots, M_r;\, g(d, M_i/d)=1} f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d_r}\right) \theta(d)$$

is equal to $(f \oplus \text{princ } \theta)\ (M_1, \ldots, M_r)$.

4. Let us write $f_1(M_1, \ldots, M_r)$ for

$$\sum_{g(d_1, \cdots, d_r)=1\,;\, d_i|M_i} f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right),$$

and g for $g(M_1, \ldots, M_r)$. Then

$$\sum_{d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) = \sum_{d|g} \sum_{g(d_1, \cdots, d_r)=d\,;\, d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

$$= \sum_{d|g} \sum_{g(\delta_1, \cdots, d_r)=1\,;\, \delta_i \left| \frac{M_i}{d} \right.} f\left(\frac{M_1}{d\delta_1}, \ldots, \frac{M_r}{d\delta_r}\right)$$

$$= \sum_{d|g} f_1\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right)$$

$$= (f_1 \cdot \text{princ E})\ (M_1, \ldots, M_r), \text{ by Theorem 3.}$$

Thus we get

THEOREM 5: If $f_1(M_1, \ldots, M_r) = \sum_{g(d_1, \cdots, d_r)=1\,;\, d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$, then

$f \cdot E = f_1 \cdot \text{princ E}$;

or, equivalently (denoting the inverse of E by E^{-1})

$$f_1 = f \cdot E \cdot \text{princ } E^{-1}$$

Corollary.—By definition* $f_1(M_1, \ldots, M_r)$

$$= \sum_{d_i|M_i} E_0\left(g(d_1, \ldots, d_r)\right) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

$$= \{E_0(g) \cdot f\}\ (M_1, \ldots, M_r), \text{ say.}$$

Therefore, taking f to be E_0 we get

$$E_0\{g(M_1, \ldots, M_r)\} = (E \cdot \text{princ } E^{-1})\ (M_1, \ldots, M_r).$$

---

* E_0 is the function which vanishes unless all the arguments are unity and then is equal to unity.

Example.—Take f to be the function $\phi_k(M_1, \ldots, M_r)$

$$= \prod_i \phi_k(M_i)$$

where $\phi_k(M)$ is the Jordan function representing the number of sets of k numbers not greater than M whose g.c.d. is prime to M. Then

$$\phi_k(M_1, \ldots, M_r) = (I_k \cdot E^{-1})(M_1, \ldots, M_r)$$

where $I_k(M_1, \ldots, M_r) = \prod_i I_k(M_i)$, $I_k(M) = M^k$,

and $E^{-1}(M_1, \ldots, M_r) = \prod_i E^{-1}(M_i)$.

Applying Theorem 5 we get

$$\sum_{g(d_1, \cdots, d_r)=1} \phi_k\left(\frac{M_1}{d_1}\right) \ldots \phi_k\left(\frac{M_r}{d_r}\right) = (I_k \cdot E^{-1} \cdot E \cdot \text{princ } E^{-1})(M_1, \ldots, M_r)$$

$$= (I_k \cdot \text{princ } E^{-1})(M_1, \ldots, M_r)$$

$$= \sum \left(\frac{M_1, \ldots, M_r}{d^r}\right)^k \mu(d)$$

$$= (M_1, \ldots, M_r)^k \sum \mu(d)/d^{rk},$$

where μ is the Mobius function and the summations on the right-side are for all common divisors d of $M_1, \ldots, M_r$.

5. Let $f(M_1, \ldots, M_r)$ be the principal function equivalent to $\psi(M)$ and consider the sum

$$\sum_{g(d_1, \cdots, d_r)=1\,;\, d_i | M_i} f_1(d_1, \ldots, d_r) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

f_1 being an arbitrary function. Obviously we need consider only such divisors which make $\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}$ equal, say, to t. Also we are to have $g(d_1, \ldots, d_r) = 1$, so that it follows that t is the g.c.d. of $M_1, \ldots, M_r$. Hence there is only one set of divisors $\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)$ for which the terms of the sum can be non-vanishing. Therefore we have

$$\sum_{g(d_1, \cdots, d_r)=1\,;\, d_i | M_i} f_1(d_1, \ldots, d_r) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) = f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right) \psi(t)$$

Taking f_1 and ψ to be E we have

$$E_0(g) \cdot \text{princ } E = E, \text{ or } E_0(g) = E \cdot \text{princ } E^{-1}$$

which is the corollary of Th. 5.

Hence we have

THEOREM 6: If t denotes the g.c.d. of $M_1, \ldots, M_r$, then

$$f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)\psi(t) = \left\{\left((f_1 \times E_0(g)\right) \cdot \text{princ}\,\psi)\right\}(M_1, \ldots, M_r)$$

$$= \left\{\left(f_1 \times (E \cdot \text{princ } E^{-1})\right) \cdot \text{princ}\,\psi\right\}(M_1, \ldots, M_r).$$

Taking $f_1 = E$ we get

$$\psi\{g(M_1, \ldots, M_r)\} = (E \cdot \text{princ } E^{-1}\,\psi)(M_1, \ldots, M_r)$$

$$= [E \cdot \text{princ}\,(\psi \cdot E^{-1})](M_1, \ldots, M_r).$$

Thus we have

THEOREM 7*: If $f(M_1, \ldots, M_r)$ be a function $\psi(t)$ of the g.c.d. t of $M_1, \ldots, M_r$, then it is the integral of the principal function equivalent to the function whose integral is ψ.

Taking ψ to be E in Theorem 6 we get

THEOREM 8: If t is the g.c.d. of $M_1, \ldots, M_r$, then

$$f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right) = \left\{\left(f \times (E \cdot \text{princ } E^{-1})\right) \cdot \text{princ } E\right\}(M_1, \ldots, M_r).$$

Analogous to Theorem 6 we can easily obtain

THEOREM 9: If t is the g.c.d. of $M_1, \ldots, M_r$, then

$$\left\{\left(f_1 \times E_0(g)\right) \oplus \text{princ}\,\psi\right\}(M_1, \ldots, M_r)$$

$$= f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)\psi(t), \text{ if } g\left(t, \frac{M_i}{t}\right) = 1$$

$$= 0, \text{ otherwise.}$$

6. We shall require the following considerations in Section 3.

Let $P(a_1, \ldots, a_r; M)$ denote the number of solutions in non-zero positive integers of the equation

$$a_1x_1 + \ldots + a_rx_r = M,$$

the a's and M being positive integers, the g.c.d. of $x_1, \ldots, x_r$ being unity. Consider the function

$$\theta(M) = \sum_{a_1 M_1 + \cdots + a_r M_r = M} f\{g(M_1, \ldots, M_r)\}.$$

Let $g(M_1, \ldots, M_r) = d$. Then d is also a divisor of M.

---

* This result, when f is multiplicative, is the theorem of R. Vaidyanathaswamy mentioned in the Introduction.

It is therefore clear that we may write

$$\theta(\mathrm{M}) = \sum_{d|\mathrm{M}} \sum_{\substack{a_1\mathrm{N}_1+\cdots+a_r\mathrm{N}_r=\mathrm{M}/d \\ g(\mathrm{N}_1,\cdots,\mathrm{N}_r)=1}} f(d)$$
$$= \sum_{d|\mathrm{M}} f(d)\,\mathrm{P}(a_1, \ldots, a_r;\ \mathrm{M}/d).$$

This relation may also be written in the form

$$(\theta \cdot f^{-1})(\mathrm{M}) = \mathrm{P}(a_1, \ldots, a_r;\ \mathrm{M}),$$

provided $f(1) \neq 0$.

Section 2. *Multiple Dirichlet Series.*

1. A series of the form

$$\sum_{\mathrm{M}_i=1\ ;\ i=1,\cdots,r}^{\infty} f(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$$

shall be called a multiple Dirichlet series. In all cases under consideration we shall suppose that $s_1, \ldots, s_r$ are so chosen as to ensure the absolute convergence of the series.

Let $\sum f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$,
$\sum f_2(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$

be two multiple Dirichlet series. Then it can readily be shown that their product is the series

$$\sum (f_1 \cdot f_2)(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r}).$$

Let f_2 be the principal function equivalent to ψ. Then

$$\sum f_2(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r}) = \sum \psi(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r}.$$

It follows that

$$\{\sum f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})\}\{\sum \psi(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r}\}$$
$$= \sum \left\{\sum_d f_1\left(\frac{\mathrm{M}_1}{d}, \ldots, \frac{\mathrm{M}_r}{d}\right)\psi(d)\right\} \Big/ (\mathrm{M}_1^{s_1} \ldots \mathrm{M}_r^{s_r})$$

where $\sum_d$ denotes summation for all common divisors d of $\mathrm{M}_1, \ldots, \mathrm{M}_r$.

2. Now $\sum_{g(\mathrm{M}_1,\cdots,\mathrm{M}_r)=1} 1/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$
$$= \sum \mathrm{E}_0\,[g(\mathrm{M}_1, \ldots, \mathrm{M}_r)]/(\mathrm{M}_1^{s_1} \ldots \mathrm{M}_r^{s_r})$$
$$= \sum (\mathrm{E} \cdot \mathrm{princ}\,\mathrm{E}^{-1})(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots \mathrm{M}_r^{s_r})$$
$$= \{\sum \mathrm{E}(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})\}\sum \mu(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r}$$
$$= \{\prod_{i=1}^{r} \zeta(s_i)\}/\zeta(\sum s_i),$$

where $\zeta(s)$ is the Riemann Zeta function $\sum \frac{1}{n^s}$.

Since $\psi[g(M_1, \ldots, M_r)] = [E_0(g) \cdot \text{princ } \psi](M_1, \ldots, M_r)$

we have also

$$[\{\Pi \zeta(s_i)\}/\zeta(\Sigma s_i)] \times \Sigma \psi(M)/M^{s_1+\cdots+s_r} = \Sigma \psi[g(M_1, \ldots, M_r)]/(M_1^{s_1}, \ldots,$$

Examples.—(1) Put $\psi = \mu(M)$. Then we get

$$\Sigma \mu\big(g(M_1, \ldots, M_r)\big)/(M_1^{s_1}, \ldots M_r^{s_r}) = \{\Pi \zeta(s_i)\}/\{\zeta(\Sigma s_i)\}^2,$$

(2) Put $\psi =$ the Jordan function ϕ_k. Then

$$\Sigma \phi_k\big(g(M_1, \ldots, M_r)\big)/(M_1^{s_1}, \ldots, M_r^{s_r}) = \frac{\Pi \zeta(s_i)}{\zeta(\Sigma s_i)} \times \frac{\zeta(\Sigma s_i - k)}{\Pi \zeta(s_i)} = \zeta(\Sigma s_i - k)/\zeta(\Sigma s_i).$$

(3) Put $\psi(M) = \sigma_a(M)$ representing the sum of the ath powers of the divisors of M. Then

$$\Sigma \sigma_a\big(g(M_1, \ldots, M_r)\big)/(M_1^{s_1} \ldots M_r^{s_r}) = \frac{\Pi \zeta(s_i)}{\zeta(\Sigma s_i)} \times \zeta(\Sigma s_i)\, \zeta(\Sigma s_i - a) = \{\Pi \zeta(s_i)\}\, \zeta(\Sigma s_i - a).$$

(4) Put $\psi(M) = \sigma_a(M)\, \sigma_b(M)$. Then

$$\Sigma \sigma_a\big(g(M_1, \ldots, M_r)\big)\, \sigma_b\big(g(M_1, \ldots, M_r)\big)/(M_1^{s_1} \ldots M_r^{s_r}) = \frac{\Pi \zeta(s_i)}{\zeta(\Sigma s_i)} \times \frac{\zeta(\Sigma s_i)\, \zeta(\Sigma s_i - a)\, \zeta(\Sigma s_i - b)\, \zeta(\Sigma s_i - a - b)}{\zeta(2\Sigma s_i - a - b)}$$

$$= \{\Pi \zeta(s_i)\}\, \zeta(\Sigma s_i - a)\, \zeta(\Sigma s_i - b)\, \zeta(\Sigma s_i - a - b)/\zeta(2\Sigma s_i - a - b).$$

3. Theorem 6 of Section 1 gives

$$\psi(t) f\Big(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\Big) = \{\big(f \times E_0(g)\big) \cdot \text{princ } \psi\}(M_1, \ldots, M_r)$$

where $t = g(M_1, \ldots, M_r)$.

Therefore, $\Sigma \psi(t) f\Big(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\Big)\Big/(M_1^{s_1}, \ldots, M_r^{s_r})$

$$= \{\Sigma f(M_1, \ldots, M_r)\, E_0\{g(M_1, \ldots, M_r)\}/(M_1^{s_1} \ldots M_r^{s_r})\} \times \Sigma \psi(M)/M^{s_1+\cdots+s_r}.$$

But $\underset{g(M_1, \cdots, M_r)=1}{\Sigma} f(M_1, \ldots, M_r)/(M_1^{s_1}, \ldots, M_r^{s_r})$

$$= \Sigma f(M_1, \ldots, M_r)\, E_0\{g(M_1, \ldots, M_r)\}/(M_1^{s_1} \ldots M_r^{s_r})$$

Hence we have $\Sigma \psi(t) f\Big(\frac{M_1}{t}, \frac{M_2}{t}, \ldots, \frac{M_r}{t}\Big)\Big/(M_1^{s_1} \ldots M_r^{s_r})$

$$= \{\Sigma \psi(M)/M^{s_1+\cdots+s_r}\} \times \underset{g,\, (M_1, \cdots, M_r)=1}{\Sigma} f(M_1, \ldots, M_r)/(M_1^{s_1} \ldots M_r^{s_r}),$$

where t is the g.c.d. of $M_1, \ldots, M_r$.

Section 3. *Multiple Lambert Series.*

The series $\sum_{\substack{m_i=1 \\ i=1,\cdots,r}}^{\infty} f(m_1, \ldots, m_r) \frac{x_1^{m_1}, \ldots, x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})}$

shall be called a multiple Lambert series. Here again, as in the previous section we shall suppose that the series considered are absolutely convergent.

Expanding the series as a power series in $x_1, \ldots, x_r$, we get

$$\Sigma f(m_1, \ldots, m_r) \frac{x_1^{m_1}\ldots x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma (f \cdot \mathrm{E})(m_1, \ldots, m_r)\, x_1^{m_1}\ldots x_r^{m_r}.$$

Replacing f by $f \cdot \mathrm{E}^{-1}$ we see that

$$\Sigma (f \cdot \mathrm{E}^{-1})(m_1, \ldots, m_r) \frac{x_1^{m_1}\ldots x^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma f(m_1, \ldots, m_r)\, x_1^{m_1}\ldots x_r^{m_r}.$$

Let $f(m_1, \ldots, m_r) = \psi\{g(m_1, \ldots, m_r)\}$,

$$= \{\mathrm{E} \cdot \mathrm{princ}(\mathrm{E}^{-1} \cdot \psi)\}(m_1, \ldots, m_r).$$

Then we get

$$\Sigma\, \mathrm{princ}(\mathrm{E}^{-1} \cdot \psi)(m_1, \ldots, m_r) \frac{x_1^{m_1}\ldots x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma \psi\{g(m_1, \ldots, m_r)\}\, x_1^{m_1}, \ldots, x_r^{m_r}.$$

The left-side is equal to

$$\sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{(x_1, \ldots, x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)}$$

so that we have

$$\sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{(x_1 \ldots x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)} = \Sigma \psi\{g(m_1, \ldots, m_r)\}\, x_1^{m_1}\ldots x_r^{m_r}.$$

Examples.—(1) Take $\psi(m) = m^k$ so that $\psi \cdot \mathrm{E}^{-1}$ is the Jordan function ϕ_k. Then we get

$$\sum_{m=1}^{\infty} \phi_k(m) \frac{(x_1\ldots x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)} = \Sigma \{g(m_1, \ldots, m_r)\}^k\, x_1^{m_1}\ldots x_r^{m_r}.$$

(2) Put $\psi = \sigma_a$, the sum of the ath powers of the divisors of the argument. Since $(\sigma_a \cdot \mathrm{E}^{-1})(m) = m^a$ we have

$$\sum_{m=1}^{\infty} \frac{m^a (x_1\ldots x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)} = \Sigma \sigma_a \{g(m_1, \ldots, m_r)\}\, x_1^{m_1}\ldots x_r^{m_r}.$$

Let us put $x_1 = x^{a_1}, \ldots, x_r = x^{a_r}$. Then we have

$$\Sigma \psi \{g(m_1, \ldots, m_r)\} x^{a_1 m_1 + \ldots + a_r m_r} = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{ma_1}) \ldots (1 - x^{ma_r})}$$

But the left-side is equal to

$$\sum_{m=1}^{\infty} \left\{ \sum_{a_1 m_1 + \cdots + a_r m_r = m} \psi \{g(m_1, \ldots, m_r)\} \right\} x^m = \sum_{m=1}^{\infty} \left\{ \sum_{d \mid m} \psi(d) \, \mathrm{P}\left(a_1, \ldots, a_r; \frac{m}{d}\right) \right\} x^m$$

where $\mathrm{P}(a_1, \ldots, a_r; m)$ is the number of solutions in non-zero positive integers of the equation

$$a_1 x_1 + \ldots + a_r x_r = m$$

the g.c.d. of $x_1, \ldots, x_r$ being unity (see Section 1 § 6).

Thus we get

$$\sum_{m=1}^{\infty} (\psi \cdot \mathrm{P})(m) x^m = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{a_1 m}) \ldots (1 - x^{a_r m})}.$$

Examples.—(1) Put $\psi = \mathrm{E}$. Then we find

$$\sum_{m=1}^{\infty} \left\{ \sum_{d \mid m} \mathrm{P}(a_1, \ldots, a_r; d) \right\} x^m = \frac{x^{a_1 + \ldots + a_r}}{(1 - x^{a_1}) \ldots (1 - x^{a_r})}.$$

(2) Put $\psi = \mathrm{E}_0$. Then

$$\sum_{m=1}^{\infty} \mathrm{P}(a_1, \ldots, a_r; m) x^m = \sum_{m=1}^{\infty} \mu(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{ma_1}) \ldots (1 - x^{ma_r})}.$$

(3) Put $\psi(m) = m^k$. Then

$$\sum_{m=1}^{\infty} \left\{ m^k \sum_{d \mid m} \mathrm{P}(a_1, \ldots, a_r; d/d^k) \right\} x^m = \Sigma \frac{\phi_k(m) x^{m(a_1 + \ldots + a_r)}}{(1 - x^{ma_1}) \ldots (1 - x^{ma_r})}.$$

In particular if we put $a_1 = \ldots = a_r = 1$, and write

$$\mathrm{P}(1, \ldots, 1; m) = \mathrm{P}_r(m),$$

we have

$$\sum_{m=1}^{\infty} \sum_{m_1 + \cdots + m_r = m} \psi \{g(m_1, \ldots, m_r)\} x^m = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{P}_r)(m) x^m = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \left(\frac{x^m}{1 - x^m}\right)^r.$$

Equating coefficient of x^m we get

$$\sum_{\substack{m_1+\ldots+m_r=m \\ m_i \geqslant 1}} \psi\{g(m_1, \ldots, m_r)\} = \sum_{d|m} \psi\left(\frac{m}{d}\right) \mathrm{P}_r(d)$$

$$= \sum_{d|m} \binom{d-1}{r-1} (\psi \cdot \mathrm{E}^{-1})\left(\frac{m}{d}\right),$$

where $\binom{n}{r}$ denotes $n(n-1)\ldots.(n-r+1)/r!$

In particular $\sum_{d|m} \mathrm{P}_r(d) = \binom{m-1}{r-1}$,

i.e., $$\mathrm{P}_r(m) = \sum_{d|m} \binom{d-1}{r-1} \mu\left(\frac{m}{d}\right);$$

and $m^k \sum_{d|m} \mathrm{P}_r(d)/d^k = \sum_{d|m} \binom{d-1}{r-1} \phi_k\left(\frac{m}{d}\right)$.

Let the series $\sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m)/m^r$ be absolutely convergent. Then as x tends to 1,

$$\sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \left(\frac{x^m}{1-x^m}\right)^r \sim \left\{\sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m)/m^r\right\} (1-x)^{-r}.$$

Therefore, if we denote

$\sum_{m_1+\cdots+m_r=m} \psi\{g(m_1, \ldots, m_r)\}$ by $\Psi(m)$, we have

$$\sum_{m \leqslant n} \Psi(m) \sim \frac{n^r}{\Gamma(r+1)} \times \Sigma (\psi \cdot \mathrm{E}^{-1})(m)/m^r *$$

$$= \frac{n^r}{\zeta(r)\,\Gamma(r+1)} \sum_{m=1}^{\infty} \psi(m)/m^r.$$

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* See Titchmarsh, *Theory of Functions*, p. 242.

BACTERIAL CHEMOTHERAPY, I: SYNTHESIS OF N^1-SUBSTITUTED SULPHANILAMIDES*

By S. Rajagopalan

(From the Haffkine Institute, Bombay)

Received April 30, 1943

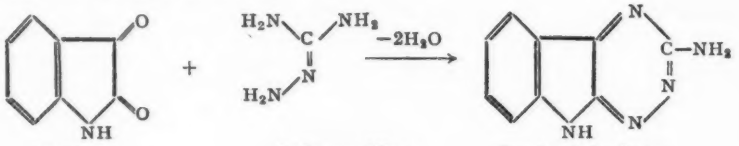
(Communicated by Lt.-Col. S. S. Sokhey, M.A., M.D., I.M.S., F.A.SC.)

Systematic syntheses in the sulphanilamide group in progress at this Institute have as their objective the discovering of hitherto unknown compounds which may be of even greater therapeutic usefulness or devoid of the deleterious effects of those drugs already in clinical usage and the collection of useful information leading to an extension of our present stock of knowledge on the mode of action of these drugs which would ultimately enable the rational evolution of more powerful members of the sulphonamide group.

The view is widely held that compounds with a higher degree or greater range of therapeutic activity than sulphanilamide should be sought among its derivatives with cyclic, particularly heterocyclic, substituents at the sulphonamide radical. The present communication is a further extension of the systematic work undertaken[2] to synthesise the above class of compounds with all feasible ring structures and study their physiological activity in relation to their chemical constitution.

The N^1-substituted sulphanilamides (Nos. 14–23) that were synthesised in this connection are listed in Table I.

The requisite starting amines were obtained by the methods reported in literature with few modifications, the only exception being 3-amino indotriazine, necessary for the preparation of the corresponding sulphonamide (No. 23). The procedure of De and Datta[4] was adopted for the synthesis in good yield of 3-amino indotriazine by condensation of isatin with amino guanidine in glacial acetic acid:



Isatin — Aminoguanidine — 3-amino indotriazine

* A preliminary note covering part of the material presented in this communication appeared in *Current Science*.[1]

TABLE I

Serial No.	Name	M.P./°C.	Nitrogen percentage	
			Found	Required
14	4-Sulphanilamido acetophenone*	189–90	9·4	9·7
15	ω-Sulphanilamido acetophenone	176–77 dec.	9·5	9·7
	ω-Sulphanilamido acetophenone hydrochloride	200–02 dec.	..	..
	ω-(N^4, Acetylsulphanilamido) acetophenone	151–52 dec.	..	..
16	ω-Sulphanilamido acetonaphthone ..	169	8·1	8·2
	ω-Sulphanilamido, α-acetonaphthone hydrochloride	189 dec., sintg. 185	..	..
	ω-(N^4, Acetyl sulphanilamido) acetonaphthone	2021-04 dec,	..	..
17	N^4-Acetyl sulphanilamido guanidine ..	117–18 dec.	25·4	25·9
18	3, N^4-Acetyl sulphanilamido, 1 : 2: 4 - triazole	210 dec.	25·1	24·9
19	5-Sulphanilamido benezotriazole	135- 37	24·0	24·2
20	5-Sulphanilamido indazole	243- 44 dec.	19·2	19·4
	5-(N^4, Acetyl sulphanilamido) indazole ..	262 dec.	..	..
21	7-Sulphanilamido indazole	249 -50 dec.	19·3	19·4
22	1-Sulphanilyl indole	159 dec.	10·1	10·3
	1-(N^4, Acetyl sulphanilyl) indole	146–47	9·0	9·0
23	3-Sulphanilamido indotriazine	200–01 dec.	24·2	24·7
	3-(N^4, Acetyl sulphanilamido) indotriazine ..	261–62 dec.	..	..
36	*m*-Hydroxy benzylidene sulphanilamide ..	138	10·3	10·1
37	*o*-Nitrobenzylidene sulphanilamide ..	indef.	13·5	13·8
38	*o*-Nitrobenzylidene sulphathiazole ..	indef.	14·0	14·4
39	*m*-Nitrobenzylidene sulphathiazole ..	220–22 dec.	13·9	14·4

* Literature[3] gives the melting point 208°C.

The amino bodies were condensed severally with crystallised acetyl sulphanilyl chloride in pyridine medium and the resulting N^4-acetyl sulphanilamide derivatives subjected to the hydrolytic action of hot dilute hydrochloric acid, with one exception, namely, No. 22, which was obtained by hydrolysis of its N^4-acetyl derivative with hot dilute (10 parts of 10 per cent. sodium hydroxide. Except in the case of the N^4-acetyl derivatives of guanidine (No. 17) and triazole (No. 18), where the attempts to isolate the finial compounds were not met with success, hydrolysis to the sulphonamides proceeded smoothly.

Examination of the Schiff's bases derived from the sulphonamides, as also those of the sulphones, has disclosed that while most of them are as active as the parent compounds themselves, at least in experimental streptococcal infections, a few of them were remarkably much less toxic.[5, 11] These compounds are chemically rather unstable and their activity is conceived[5, 6, 12] to be due to their cleavage into the original sulphonamides, on oral administration. While this mechanism of action holds for most members of this group, there are a few instances for which this explanation would not appear

acceptable without further modification. Thus, for example, *p*-hydroxy benzylidene sulphanilamide[7, 8] is said to be more active than sulphanilamide itself. Again, the high activities claimed for the N^4—*p*, nitrobenzylidene sulphonamides[9, 10, 11] are noteworthy. It is also of interest that both *o*-nitrobenzylidene and *m*-hydroxy benzylidene sulphapyridines[8, 9, 11] should be rated superior to sulphapyridine or sulphanilamide against streptococci, but inferior to sulphapyridine against pnemococci. The results with the sulphapyridine derivatives are remarkable: while a decreased activity towards the pnemococcus is conceivable, it is difficult to understand how sulphapyridine, if it is the active agent, could give an *increased* activity against streptococci, when administered as compounds which liberate it in the body. However, if these results are true, the only explanation that could be advanced would be that the " aldehyde half " of the molecule or its derivative elaborated *in vivo* exerts a modifying action on the chemotherapeutic efficacy of the " active half ". The Schiff's bases (Nos. 36–39) shown in the table, besides a few others already reported in literature, were therefore prepared in order to investigate in some detail the mechanism of action of Schiff's bases derived from the sulphonamides.

The compounds reported herein are being investigated as to their usefulness in experimental bacterial infections with particular reference to plague in mice at this Institute.

Experimental

The synthesis of the final sulphonamides having been effected according to the customary procedures, the experimental details have been omitted.

3-*Amino indotriazine*.—A mixture of aminoguanidine carbonate 2·74 gm.) and isatin (2·94 gm., 1 mol.) in glacial acetic acid (100, c.c.) was gently refluxed for one half hour, diluted with water and basified with excess of ammonia. The 3-amino indotriazine recrystallised from alcohol as yellow needles, melting at 195–96°; yield-4·5 gm. (Found: N, 37·3; $C_9H_7N_5$ requires N, 37·8 %.)

The *Schiff's bases* (Nos. 36–39) were prepared by boiling together alcoholic solutions of molecular proportions of the appropriate aldehyde and the sulphonamide until crystal-separation occurred. The reaction mixture was cooled, the separated solid filtered and washed free of any traces of unreacted initial compounds by means of alcohol. These could not be recrystallised from solvents owing to their instability; they consisted of pale plates.

Except Nos. 17 and 18, the sulphonamides (Nos. 14–16, 19–23) were all recrystallised from alcohol; Nos 17 and 18 were obtained from water, all of

them (Nos. 14–22) were obtained as characteristic colourless needles, the exception being No. 18 which formed colourless prisms. The indotriazine derivative consisted of yellow needles.

The yields of all the compounds were good.

The author's grateful thanks are due to Lt.-Col. Sokhey for his kind interest and the Lady Tata Memorial Trust for the award of a Research Scholarship.

Summary

With the object of ascertaining the relative antibacterial merits of sulphonamides derived, among a few others, chiefly from heterocyclic ring systems, a series of N^1-substituted sulphonamides containing the triazole, benzotriazole, indazole, indole, and indotriazine rings have been synthesised. Additional compounds reported are the sulphonamide derivatives of acetophenone and guanidine.

A few Schiff's bases derived from the sulphonamides have also been prepared for their possible therapeutic usefulness and for a study of the mode of action of the sulphonamido Schiff's bases in general.

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BACTERIAL CHEMOTHERAPY, II : SYNTHESIS OF POSSIBLE INTESTINAL ANTISEPTICS OF THE SULPHANILAMIDE GROUP

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(Communicated by Lt.-Col. S. S. Sokhey, M.A., M.D., I.M.S., F.A.SC.)

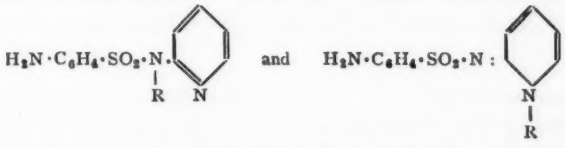
The successful application of the sulphonamides to the chemotherapy of many types of bacterial infections led to their trial in infections of the Cholera-dysentery-typhoid group, which are characterised by their main or entire localisation in the intestinal tract. The prerequisites of a compound to be an ideal intestinal antiseptic are that it should be poorly absorbed and also possess a strong bacteriostatic and bactericidal local action. The properties of such a compound would permit the attainment of a high concentration in the intestine without being too toxic, due to a low concentration in the blood and tissues. Sulphaguanidine, a drug fulfilling these specifications, has been introduced by Marshall *et al.*[1] and heralds an advance in intestinal antisepsis. Although sulphapyridine and sulphathiazole were found on clinical trial to have some curative value in bacillary dysentery, these drugs have the disadvantage, as far as intestinal infections are concerned, of being readily absorbed from the bowels and perhaps of being renal irritants in dehydrated patients. However, the evolution of succinyl sulphathiazole[2]—a drug which is believed to owe its activity to its slow cleavage into sulphathiazole in the intestine—should be considered a right step in this direction.

Three compounds of the sulphonamide group—sulphathiazoline, sulphapyridine and sulphathiazole—were therefore chosen for further synthetic elaboration and study of the resulting products as to their usefulness in intestinal as well as a few other bacterial infections. It appeared that by introducing substituents at the sulphonamide part of the molecules of the sulphonamido heterocycles, compounds insoluble in alkali and therefore unlikely to be absorbed from the intestine, which is alkaline in pH, could be obtained. A list of some of the compounds prepared with this object are listed.

Serial No.	Name	M.P./°C.	Nitrogen percentage	
			Found	Required
27	3-Methyl. 2-sulphanilimido, 2 : 3-dihydro-thiazoline	196–98	15·4	15·5
28	3-Ethyl. 2-sulphanilimido, 2 : 3-dihydro-thiazoline	181–82 dec.	14·6	14·7
29	N^1-Phenyl sulphathiazoline	glassy liquid at 156, which clears at ca. 185	12·6	12·6
	N^4-Acetyl, N^1-phenyl sulphathiazoline ..	230 —	..	..
30	N^1-Allyl sulphathiazoline	186–89, softg. 187	13·8	14·1
	N^4-Acetyl, N^1-allyl sulphathiazoline ..	179–81, sintg. 176	..	..
32	1-*p*, Nitrobenzyl, 2-sulphanilimido, 1 : 2-dihydro pyridine	234 dec.	14·4	14·6
	1-*p*, Nitrobenzyl-2-acetyl sulphanilimido, 1 : 2-dihydropyridine	215–18	..	..
33	1-*p*, Nitrobenzyl-2-(*p*, nitrobenzylamino)-benzene sulphonilimido, 1 : 2-dihydro-pyridine	208–10	13·3	13·3
34	3-*p*, Nitrobenzyl-2-sulphanilimido, 2 : 3-dihydrothiazole	199–200 dec.	13·8	14·4
35	3-*m*, Nitrophenacyl-2-sulphanilimido, 2 : 3-dihydrothiazole	238–39	13·4	13·4
	3-*m*, Nitrophenacyl-2-acetyl-sulphanilimido, 2 : 3-dihydrothiazole	216–18	..	..

The nitro aralkyl derivatives (Nos. 32–35) were selected because of the high activities reported for a somewhat similar compound, N,[4] 4′-nitrobenzyl sulphanilamide.[3]

There are two alternative structures possible for the compounds obtained by the above kind of operation on the sulphonamides. Taking the example of one of the sulphonamido heterocycles, namely, sulphapyridine, the introduction of an alkyl radical (R) can furnish the following isomeric products.



The production of one or other of the isomeric forms are conditioned by the experimental technique adopted and have recently been well investigated. The nomenclature adopted for the compounds reported herein are, therefore, in accordance with these studies.[4,5]

Experimental

3-*Methyl- and* 3-*ethyl*, 2-*sulphanilimido*, 2 : 3-*dihydrothiazolines* (Nos. 27 and 28) were obtained by the action of a slight excess of the appropriate alkyl

sulphate or iodide on an aqueous alkaline solution of sulphathiazoline. They consisted of colourless needles or plates after crystallisation from alcohol.

N^1-Phenyl sulphathiazoline (No. 29): *2-Anilino thiazoline.*—The method available in literature[6] gave only a poor yield of this compound. A modified procedure afforded the desired amine in a very much better yield; a mixture of β-bromoethylamine hydrobromide (10 gm.) in water and phenyl *iso*thiocyanate (6·7 gm., 1 mol.) was cooled and treated with excess of 20 per cent. sodium hydroxide. The separated crystals were purified through a dilute hydrochloric acid solution and separated from alcohol in colourless needles, m.p. 158–60°.

N^1-Phenyl sulphathiazoline was prepared by the acid hydrolysis of its N^4-acetyl derivative, in turn obtained by condensation of acetyl sulphanilyl chloride with anilino thiazoline in pyridine according to general procedure.

N^1-Allyl sulphathiazoline (No. 30): *2-Allylamino thiazoline*, prepared by the same procedure as adopted for 2-anilino thiazoline, was obtained as an oil which was used as such without further purification.

The sulphonamide (No. 30) was prepared as usual from the above amine.

1-p, Nitrobenzyl-2-(p, nitrobenzylamino-) benzene sulphonimido, 1 : 2-dihydro pyridine.—A mixture of equimolecular amounts of sodium sulphapyridine and *p*-nitrobenzyl bromide in alcohol was refluxed until a neutral reaction resulted. The desired compound, which is insoluble in both dilute hydrochloric acid and sodium hydroxide, was separated from the accompanying basic monosubstitution product (No. 32) and from unreacted sulphapyridine.

The compounds (Nos. 32, 34, 35) were obtained by the acid hydrolysis of the corresponding N^4-acetyl derivatives, which in their turn were prepared by the interaction of the respective N^1-sodio sulphonamides and the appropriate alkyl bromides in alcoholic solution.

The nitro compounds (Nos. 32–35) were obtained as yellow crystals from pyridine.

The author desires to express his gratefulness to Lt.—Col. Sokhey for his keen interest and encouragement and to the Lady Tata Memorial Trust for the award of a Scholarship.

Summary

Chiefly by the action of alkyl or aralkyl halides or the alkyl sulphates on the sulphanilamido heterocycles, a series of compounds, insoluble in alkali, has been obtained. Because of the likelihood of these not being absorbed from the intestine, which is alkaline in pH, the type of compounds synthesised are expected to prove particularly useful in infections of the intestinal tract.

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BACTERIAL CHEMOTHERAPY, III: SYNTHESIS OF POSSIBLE LIPOPHILIC CHEMOTHERAPEUTICALS OF THE SULPHANILAMIDE GROUP*

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(Communicated by Lt.-Col. S. S. Sokhey, M.A., M.D., I.M.S., F.A.Sc.)

Although the compounds of the sulphanilamide group already evolved have provided medical science with some of the most potent weapons for the effective conquest of many diseases of bacterial origin, there is still a long list of bacterial infections uninfluenced by the newer chemotherapeuticals; among the latter, leprosy and tuberculosis continue to constitute the major posers to chemotherapy. This justifies further exploratory work both on compounds closely related to the effective substances already known and on their structural allies but with other substituents. The present study is one of a series of such exploratory investigations in the sulphanilamide group undertaken at this Institute with the object of extending their range of therapeutic usefulness in bacterial infections, also including those caused by the acid-fast mycobacterium.

The newer reports of the effectiveness of the treatment of experimental tuberculosis in the guinea pig and the rabbit with members of the sulphanilamide group are still discordant. A number of workers have claimed that sulphanilamide,[2, 3] sulphapyridine,[4] prontosil soluble,[5] sulphathiazole[6, 7] and sulphadiazine[7] in high blood-levels may modify the course of experimental tuberculosis. However the results obtained so far do not appear to warrant clinical trial of any of the aforementioned compounds. The low antitubercular activity of the sulphonamides, notwithstanding their direct action on many pathogenic organisms,[8] has been put down as perhaps partly due to their inability to penetrate the lipoid layer of the bacteria. Consequently, an attempt has been made to achieve better results by introducing lipophilic radicals into the sulphonamides. On this basis, only a single compound, N^1-dodecanoyl sulphanilamide, has been prepared[9] and tested experimentally without, however, consistent success.[10, 11] On the ground that sulphathiazole is more active than sulphanilamide against most bacteria, the introduction of a lipophilic substituent in the heterocyclic part of the

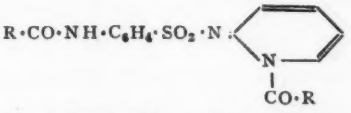
* A preliminary note covering part of the material embodied in this communication appeared in *Current Science*.[1]

former has been tried; a series of alkyl sulphathiazoles[12] as well as several sulphathiodiazoles[13] have been prepared but have not yet been tested against the tubercle bacillus. Thus the results secured so far can in no sense be called uniform.

Study of the numerous acyl derivatives of the sulphonamides hitherto synthesised has disclosed a few members effective in combating experimental coccal infections in mice associated with a low order of toxicity; some of them have also passed actual clinical trials.[14] However, with the exception of N^1-dodecanoyl sulphanilamide which is uncertainly reported, on[9, 10, 11] and the chaulmoogryl derivative[11] of sulphanilamide, none of the fatty acid derivatives of the sulphonamides appear to have been investigated as to their efficacy in tuberculosis or leprosy.

The present communication represents one of the many possible lines of attack for ascertaining whether or not the sulphonamides can be pressed to service in the conquest of mycobacterial infections by suitable synthetic operations on the molecule of sulphanilamide. Accordingly, a series of acyl derivatives of the sulphonamides have been synthesised in the expectation that these might exhibit a much greater ability to penetrate the lipoid layer of the tubercle and lepra bacilli as compared to the parent sulphonamides. The sulphonamides chosen for necessary modifications of their molecules in order to render them lipophilic were all previously known and reported to be active at least in some coccal infections. The final sulphonamides, prepared in this connection, are represented in Tabe I.

For purposes of convenience, the nomenclature of Crossley, Northey and Hultiquist[17] has been adopted. This does not however, exclude the alternative structure



for the N^4, N^1-diacyl sulphapyridines (Nos. 66–71 and 73) which is also possible.[18] In all probability they are equilibrium mixtures of the two isomeric forms. The compounds Nos. 86–92 have been assigned the structure represented in the table on the basis of their mode of preparation and in accordance with recent studies[18] on this type of compounds.

Besides the simple normal fatty acyl derivatives, a few branched-chain acyl sulphonamides (Nos. 71, 73, 79, 80, 84), of which Nos. 71, 79 and 84 contain the advantageous cyclohexyl group,[19] were also included in this study in view of the interesting results obtained by Robinson and Birch[20] with acids of the latter type.

TABLE I

Serial No.	Name	M.P./°C.	Nitrogen percentage	
			Found	Required
40	N^4-*n*, Caprylyl sulphanilamide*	198	9·6	9·7
41	N^4-Cyclohexoyl sulphanilamide	238	9·8	9·9
61	N^4-*n*, Butyryl sulphapyridine	206	12·9	13·2
63	N^4-*n*, Heptoyl sulphapyridine	193	11·6	11·6
64	N^4-*n*, Caprylyl sulphapyridine	213–14	10·8	11·2
66	N^4, N^1-Diacetyl sulphapyridine	194	11·8	12·6
67	N^4, N^1-Di (*n*, butyryl)-sulphapyridine ..	163	10·7	10·8
68	N^4, N^1-Di (*n*, caproyl)-sulphapyridine ..	155-57	10·1	9·5
69	N^4, N^1-Di (*n*, caprylyl)-sulphapyridine ..	135	8·5	8·4
70	N^4, N^1-Dibenzoyl sulphapyridine	217	9·2	9·2
71	N^4, N^1-Dicyclohexoyl sulphapyridine ..	193–95	8·6	9·0
73	N^4, N^1-Dicinnamoyl sulphapyridine ..	196–98	8·6	8·3
74	N^4-*n*, Butyryl sulphathiazole	244–46 dec.	12·4	12·5
75	N^4-*n*, Caproyl sulphathiazole	198–99	11·2	11·6
76	N^4-*n*, Heptoyl sulphathiazole	202–03	10·8	11·1
77	N^4-Palmityl sulphathiazole	140–47	8·3	8·5
78	N^4-Stearyl sulphathiazole	148–50	7·8	8·1
79	N^4-Cyclohexoyl sulphathiazole	222–23 dec.	11·5	11·5
80	N^4-Furoyl sulphathiazole	dec. above 240	12·0	12·0
81	N^4-*n*, Butyryl sulphathiazoline	224–25	12·1	12·5
82	N^1-*n*, Caproyl sulphathiazoline	181–82	11·2	11·5
83	N^4-*n*, Heptoyl sulphathiazoline	175–76	11·2	11·1
84	N^4-Cyclohexoyl sulphathiazoline	220	11·5	11·4
85	N^4-*n*, Caprylyl sulphanildimethylamide ..	79–82	8·4	8·6
86	1-Methyl, 2-(N^4-*n*, butyryl sulphanilimidio) 1 : 2-dihydro pyridine	213	12·6	12·6
87	1-Methyl, 2-(N^4-*n*, caproyl sulphanilimido) 1 : 2-dihydro pyridine	213–15	11·8	11·6
88	3-Methyl, 2-(N^4-*n*, caproyl sulphanilimido) 2 : 3-dihydro thiazole	215	11·5	11·4
89	3-Methyl, 2-(N^4-*n*, heptoyl sulphanilimido) 2 : 3-dihydro thiazole	173–74	11·3	11·0
90	3-Methyl, 2-(N^4-*n*, caprylyl sulphanilimido) 2 : 3-dihydro thiazole	153–54	10·4	10·6
91	3-Methyl, 2-(N^4-*n*, caproyl sulphanilimido) 2 : 3-dihydro thiazoline	201–03	11·1	11·4
92	3-Methyl, 2-(N^4-*n*, heptoyl sulphanilimido) 2 : 3-dihydro thiazoline	170	11·0	11·0
93	N^4-*n*, Butyryl, N^1-(*p*-nitrophenyl)-sulphanilamide	248–50 dec.	11·2	11·6
94	N^4-*n*, Caproyl, N^1-(*p*-nitrophenyl)-sulphanilamide†	152	10·3	10·7
95	N^4-*n*, Butyryl sulphanilyl sulphanilamide ..	235–36	10·6	10·6
96	N^4-*n*, Caproyl sulphanilyl sulphanilamide ..	184–86	9·4	9·9
	2-Sulphanilamido benzoic acid‡	ca. 215 dec.	9·4	9·6
97	2-(N^4-*n*, Butyryl sulphanilamido)-benzoic acid	226–28 dec.	7·3	7·7
	4-Sulphanilamido benzoic acid§	181–82	9·3	9·6
98	4-(N^4-*n*, Butyryl sulphanilamido)-benzoic acid	224–26	7·4	7·7

* Variously reported[15] to melt at 189° and 200°C.

† Literature[16] gives the m.p. 225°C.

‡ Reported by Kolloff[21] to melt at 225°C.

§ Kolloff[21] reports m.p. 202°C., Crossley *et al*[17] give m.p. 198–200°C.

Experimental

The N^4-acyl sulphonamides (Nos. 40, 41, 61, 63, 64, 74–84, 93–98) were prepared by condensation of molecular proportions of the requisite acid chlorides on the respective sulphonamides in the presence of pyridine. The condensation products, obtained by dilution of the reaction mixtures with excess of water, were severally purified through precipitation from their dilute sodium hydroxide solutions (decolorising carbon) by acidification with excess of dilute hydrochloric acid. They were finally recrystallised from either alcohol (Nos. 40, 41, 61, 63, 74–79, 93–98) or acetic acid (Nos. 64, 80–84) in colourless needles with the exception of Nos. 74, 75, 93 and 94, which were obtained as slightly pale plates or prismatic needles. The compound No. 97 was recrystallised from acetone in colourless needles.

Of the remaining N^4-acyl sulphonamides, the compounds, Nos. 85, 88–90 were prepared by operation of molecular proportions of the appropriate acid chlorides on the respective sulphonamides in pyridine medium; for the remaining compounds, *viz.*, Nos. 86, 87, 91, 92, the corresponding N^4-acyl but N^1-unsubstituted sulphonamides constituted the starting materials: aqueous alkaline solutions of these, on methylation with dimethyl sulphate, furnished the desired compounds. The compounds were first purified and freed from traces of the initial parent sulphonamides by suitable methods dictated by their differences in character when they were obtained as well defined, colourless crystals. The compound No. 85 was crystallised from acetone; the sulphathiazole derivatives, Nos. 88, 89 and 90, were obtained from acetic acid while the rest were crystallised from alcohol.

The N^4-N^1-disubstituted sulphapyridines (Nos. 66–71, 73) resulted directly by the operation of slightly more than 2 mols. of the appropriate acid chloride on sulphapyridine itself in pyridine solution. These products were isolated by taking advantage of their characteristic insolubility in both dilute mineral acids and alkalies. As hot polar solvents had the tendency to split off the acyl groups at the N^1-nitrogen of these compounds—in the case of the straight chain acid compounds, hot water broke them down to the corresponding N^4-acyl derivatives and the acids, while hot alcohol readily converted them to a mixture of the respective N^4-acyl derivatives and the acid esters—it was found advantageous to effect their recrystallisation by dilution of their solutions (charcoal) in acetone. These were then obtained as colourless plates and were dried *in vacuo* over anhydrous calcium chloride.

The yields of the final compounds reported herein, in all the instances, were good.

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Summary

On the ground that the sulphonamide group of drugs after suitable synthetic modifications of their molecules may probably be useful in mycobacterial infections, the possibility of rendering a few members, previously known to be active in coccal infections, lipophilic by the introduction of fatty acid residues in the latter has been explored. Accordingly, normal and branched chain acyl derivatives, thirty-seven in all, of sulphanilamide, sulphapyridine, sulphathiazole, sulphathiazoline, etc., have been synthesised.

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NEW CONVERGENCE AND SUMMABILITY TESTS FOR FOURIER SERIES

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1. Let $f(x)$ be a periodic summable function and

$$\psi_{x_0}(t) = |f(x_0+t)+f(x_0-t)-2f(x_0)|. \qquad (1\cdot1)$$

Theorem A of this note is a generalization of the FeJer-Lebesgue-Hardy theorem which (latter) is:—

$$\frac{1}{u}\int_0^u \psi_{x_0}(t)\,dt \to o \text{ as } u \to o \text{ implies the (C, } r)\cdot(r>0)\cdot\text{convergence of } (1\cdot2)$$

the Fourier series of $f(x)$ at x_0 to $f(x_0)$. [Zygmund[1], FeJer[2], Lebesgue[3], Hardy[4].] The subsidiary theorem B of this note discusses the relation between the various forms of the hypothesis of theorem A, and in particular their relation to the (F. L. H.) hypothesis in (1·2); and with the examples given below it in 3, establishes that the (F. L. H.) theorem is a particular case of theorem A.

Theorem C is a generalization of convergence test of the Fourier series of $f(x)$ at a point x_0 due to Hardy and Littlewood[5] (1932), the latter test being as follows:

(1) $\psi_{x_0}(t)\log t \to 0\ t \to 0$. (2) the Fourier coefficients of $f(x) \leqslant An^{-\delta}$ (1·3) ($A>0$, $\delta>0$). Hardy and Littlewood have established their test by using the Tauberian theorem associated with Valeron[6] (1917) means. In a recent paper[7] (1943) I have derived this test by proving

(*a*) (1) of (1·3) implies that $\frac{1}{\operatorname{Log} n}\sum_{r=0}^{n}\frac{S_r(x_0)}{n+1-r} \to f(x_0)$ as $n \to \infty$.

(*b*) (2) of (1·3) is exactly the Tauberian condition associated with convergence by the means in (*a*). Proceeding along the lines of my paper,[7] I have derived theorem C, from theorem A and a new Tauberian theorem in another paper[8] of mine to be published shortly. It is noteworthy that the form of theorem A permits us to replace (1) in the Hardy-Littlewood

test by the slightly broader one (1)′ $\frac{1}{u}\int_0^u \psi_{x_0}(t)\,\mathrm{Log}\,t\,dt \to 0,\ u \to 0$

or the still broader one (1)″ $\frac{1}{\mathrm{Log}\,u}\int_u^\delta \psi_{x_0}(t)\,\frac{\mathrm{Log}\,t}{t}\,dt \to 0\ u \to 0$

In 7 I have discussed particular cases of interest of A and C.

[*Note.*—In what follows "F.L.H. hypothesis" will stand for the condition in (1·2) namely $\frac{1}{u}\int_0^u \psi_{x_0}(t)\,dt \to 0$ as $u \to 0$ and F.L.H. theorem, the theorem of (1·2).]

2. *Notations and Definitions.*—The b_n, b_n', B_n, B_n'. $\psi_{x_0}(t)$, B_t, b_t, $\mathrm{S}_n(x_0)$, are defined as follows :

(1) $b_n > 0.$, (2) $b_0 \geqslant b_1 \geqslant b_2 \ldots \geqslant b_n$;

(3) $\mathrm{B}_n = \sum_{r=0}^{n} b_r$ (and $\to\infty$ as $n \to \infty$). (4) $b_t = b_{[t]}$, $\mathrm{B}_t = \mathrm{B}_{[t]}$, b_n', B_n', etc., being restricted by the same conditions;

(5) $\psi_{x_0}(t)$ defined as in (1·1); (6) $\mathrm{S}_n(x_0) = \frac{1}{2}a_0 + \sum_1^n (a_r \cos r\,x_0 + b_r \sin r x_0)$ a_r, b_r, being the Fourier coefficients of $f(x)$, $f(x)$ being as in (1·1).

3. *Statement of theorems A and B and remarks.*—

THEOREM A: If $\frac{1}{\mathrm{B}_x}\cdot\int_{a\to\infty}^{a} \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{\mathrm{B}_t}{t}\,dt \to 0 \qquad (a > 0)$ (3·1)

and $\frac{1}{\mathrm{B}_{N'}}\sum_{n_0}^{N}\frac{b_n}{\mathrm{B}_n}\cdot\mathrm{B}_n'$ is bounded for all N (3·2)

then $\frac{1}{\mathrm{B}_n'}\cdot\sum_0^n b_r'\,\mathrm{S}_{n-r}(x_0) \to f(x_0)$, as $n \to \infty$. (3·3)

Particular case of A of interest is as follows :

Let $\theta(t) = \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{\mathrm{B}_t}{t b_t}$.

THEOREM A_1:

If (*a*) $\theta(t) \to 0$ or (*b*) $\frac{1}{t}\int_0^t \theta(u)\,du \to 0$ as $t \to \infty$ then (3·3) will be true. (3·4)

[It is to be noted that (*b*) of (3·4) could be given in another form completely equivalent to it namely $\frac{1}{t}\int_0^t \theta\left(\frac{1}{u}\right) du \to 0$ as $t \to 0$; also it can be shewn that (3·4) (*a*) implies (3·4) *b*, and (3·4) *b* implies (3·1)].

Subsidiary theorem B.

If $\frac{1}{B_N'}\sum_{n_0}^{N}\frac{b_r}{B_r}B_r'$ is bounded for all $N(\geqslant n_0)$ (*i.e.* $\leqslant$ some positive k) (3·5)

then $\frac{1}{B_x}\int_{a\to\infty}^{a}\psi_{x_0}\left(\frac{1}{t}\right)\frac{B_t}{t}\,dt \to 0$ $(a>0)$ implies

$$\frac{1}{B_x'}\int_a^a \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{B_t'}{t}\,dt \to 0; \qquad (3\cdot6)$$

If (3·5) is not true (3·6) is not necessarily true. (3·7)

[*Note* 1.—It is to be noticed that (F. L. H.) hypothesis can be put in the equivalent form $\frac{1}{x}\int_a^a \psi_{x_0}\left(\frac{1}{t}\right)dt \to 0$ $x \to \infty$; so that from conditions in 2 about b_n, it follows from theorem B that hypothesis $\frac{1}{B_x}\int_a^a \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{B_t}{t}\,dt \to 0$ implies the (F. L. H.) hypothesis and if $\frac{1}{B_N}\sum_{n_0}^{N}\frac{B_r}{r}$ is bounded for all $N \geqslant n_0$ the converse is also true; so that in this case (F. L. H.) hypothesis is equivalent to hypothesis (3·1). But the form of theorem A allows us to enlarge the field of (C, *r*) summability for the Fourier series of $f(x)$ at x_0 as the following example shews:—

Example 1.—Let $0 < \rho_1 < \rho_2$ and $\rho_1 + \rho_2 < 1$ and $\psi(x) = \log\log\log x$. Let $F_n = n^{\rho_2+\rho_1\cos\psi(n)}$ and $F_n - F_{n-1} = b_n$ for sufficiently large n, say $n \geqslant n_0$. b_0, b_1, b_{n_0-1} being defined suitably to suit (1) and (2) of 2. then it can be proved that (*a*) b_n and $B_n = \sum_0^n b_r$ satisfy (1), (2), (3) of 2, and (*b*) $\frac{1}{B_N}\sum_{n_0}^{N}\frac{B_r}{r}$ is bounded for all $N \geqslant n_0$. so that from theorem A it follows $\frac{1}{B_n}\sum_0^n b_r S_{n-r}(x_0) \to f(x_0)$ if (F. L. H.) hypothesis is true. It is to be noted that $\frac{\log B_n}{\log n}$ in this case oscillates between $\rho_1+\rho_2$ and $\rho_2-\rho_1$.] (3·8)

[*Note* 2.—Hypothesis (3·2) namely boundedness of $\frac{1}{B_N'}\sum_{m_0}^{N}\frac{b_r}{B_r}B_r'$ implies $\frac{\text{Log } B_n}{\text{Log } B_n'}$ is bounded above, in particular $\frac{1}{B_N}\sum^{N}\frac{B_r}{r}$ implies $\frac{\text{Log } B_n}{\text{Log } n}$ is bounded above for $n \geqslant 2$.] (3·9)

[*Note* 3.—The following example reveals the importance of condition (3·2) in theorem A.

Example 2.—Let $f(t)$ be an even periodic bounded summable function and let its Fourier coefficients be such that Lim sup $\frac{\bar{S}_n(0)}{\text{Log } n} > 0$

where $S_n(0) = \frac{1}{2}a_0 + \sum_1^n a_r$.

Let $\psi(t) = \frac{f(t)}{|\text{Log } |t||}$ in $0 \leqslant |t| \leqslant \delta < 1$

$= 0$ otherwise.

In this case the Fe Jer theorem at $t=0$ about $\psi(t)$ is true.

Taking $b_n = \frac{1}{n+1}$ it is easy to see that $\frac{1}{B_N}\sum^{N}\frac{B_r}{r}$ in this case $\to\infty$ as $N\to\infty$.

It can be established that $\dfrac{\sum_0^n \frac{S_{n-r}(0)}{1+r}}{\sum_0^n \frac{1}{1+r}}$ does not converge to $\psi(0)=0$,

$S_n(0)$ = the sum of the 1st $\overline{n+1}$ Fourier coefficients of $\psi(t)$. [Incidentally this example establishes (3·7) of B, *via* A.] (3·9)

[*Note* 4.—In view of theorem B, it is sufficient to establish theorem A in the following form:—

$\bar{A}$. Hypothesis (3·1) implies $\frac{1}{B_n}\sum_0^n b_r S_{n-r}(x_0) \to f(x_0)$.] (3·10)

4. *Proof of theorem B.*—

$$\text{Let } a_n = \int_n^{n+1} \psi_{x_0}\left(\frac{1}{t}\right) dt. \tag{4·1}$$

It can be easily established that

$\frac{1}{B_n}\cdot\int_0^{\infty}\psi_{x_0}\left(\frac{1}{t}\right)\frac{B_t}{t}\,dt \to 0$ as $x\to\infty$ is completely

equivalent to $\frac{1}{B_n}\sum_{n_0}^{n}\frac{B_r a_r}{r}\to 0$ as $n\to\infty$. (4·2)

and similarly with regard to $\frac{1}{B_x'}\int_a^{\pi} \psi_{x_0}\left(\frac{1}{t}\right)\frac{B_t'}{t}\,dt \to 0$ (4·2)

so that we will have established theorem B if we prove the following:

$\overline{\mathbf{B}}$. If $x_n = \frac{1}{B_n}\sum_{n_0}^{n}\frac{B_r a_r}{r}$, and $y_n = \frac{1}{B_n'}\sum_{n_0}^{n}\frac{B_r' a_r}{r}$, and

$\frac{1}{B_n'}\sum_{n_0}^{n}\frac{b_r}{B_r}B_r'$ is bounded, then $x_n \to 0$ implies $y_n \to 0$. (4·3)

Without the boundedness condition it (4·3) is not necessarily true (4·4)

Proof of $\overline{B}$.—From definition of x_n in $\overline{B}$, we have

$$a_n = n(x_n - x_{n-1}) + \frac{n b_n}{B_n} x_{n-1}$$

and $y_n = \frac{1}{B_n'}\cdot\sum_{n_2}^{n} B_r'(x_r - x_{r-1}) + \frac{1}{B_n'}\cdot\sum_{n_0}^{n}\frac{b_r}{B_r}\cdot B_r'\cdot x_{r-1}$

$= y_{n,1} + y_{n,2}$. (4·5)

From the boundedness of $\frac{1}{B_n'}\cdot\sum_{0}^{n}\frac{b_r}{B_r}B_r'$, and $x_n \to 0$ and hypothesis in, 2, about B_n', it follows that

$$y_{n,2} \to 0 \text{ as } n \to \infty. \tag{4·6}$$

From the fact that B_n' increases monotonically to ∞ and $x_n \to 0$ it follows from Kroneker's theorem that

$$y_{n,1} \to 0 \text{ as } n \to \infty \tag{4·7}$$

thus (4·5), (4·6) and (4·7) establish (4·3) of $\overline{B}$.

(4·4) is covered by example 2 in (3·9); thus establishing $\overline{B}$.

5. *Proof of* $\overline{A}$.—By the usual formula in Fourier series and Riemann-Lebesgue Lemma, we have

$$S_n(x_0) - f(x_0) = \frac{1}{2\pi}\int_0^{\delta}\{f(x_0+t) + f(x_0-t) - 2f(x_0)\}\frac{\sin(n+\frac{1}{2})t}{\sin t/2}\,dt + \epsilon(n,\delta) \tag{5·1}$$

where $0 < \delta < \pi$, δ to be suitably chosen and fixed and $\epsilon(n,\delta) \to 0$ $n \to \infty$.

Hence $\left|\frac{1}{B_n}\sum_{0}^{n} b_r S_{n-r}(x_0) - f(x_0)\right| \leqslant \frac{1}{2\pi B_n}\cdot\int_0^{\delta}\psi_{x_0}(t)\cdot\frac{|k_n(t)|}{\sin\frac{t}{2}}\,dt + \epsilon'(n,\delta)$ (5·2)

where $k_n(t) = \sum_0^n b_r \sin\left(n+\frac{1}{2}-r\right)t$ and

$$\epsilon'(n,\delta) = \left|\sum_0^n \frac{b_r\,\epsilon(n-r,\delta)}{B_n}\right| \to 0 \text{ as } n\to\infty.$$

We shall here prove

$$\left.\begin{aligned} &(a)\quad \frac{|k_n(t)|}{\sin\frac{t}{2}} \leqslant (2n+1)\,B_n \\ &(b)\quad \frac{|k_n(t)|}{B_{\frac{1}{t}}} \leqslant k_1 \text{ (some positive constant) in } \frac{1}{n} \leqslant t \leqslant \delta \end{aligned}\right\} \quad (5\cdot3)$$

(*a*) follows from $\left|\frac{\sin mz}{\sin z}\right| \leqslant m$ for integral m.

Proof of 5·3 (*b*).

$$-\sum_0^n b_r \sin\left(n+\frac{1}{2}-r\right)t$$

$$= \text{Imaginary part of } \left\{e^{-i\left(n+\frac{3}{2}\right)t} \cdot \sum_0^n b_r\, e^{i(r+1)t}\right\}$$

and $\sum_{r=0}^{n} b_r\, e^{i(r+1)t} = \sum_{r=0}^{\left[\frac{1}{t}\right]} + \sum_{\left[\frac{1}{t}\right]+1}^{n} \quad \left(\frac{1}{n} \leqslant t\right)\cdot$

$$= A_{n,1} + A_{n,2}.$$

By hypothesis in 2 about b_n and Abel's inequality we have

$$|A_{n,2}| \leqslant \frac{k}{t}\cdot b_{\frac{1}{t}} \leqslant k'\, B_{1/t};$$

and

$$|A_{n,1}| \leqslant \cdot\, B_{1/t}.$$

Hence $|k_n(t)| \leqslant k_2 \cdot B_{1/t}$ in $\frac{1}{n} \leqslant t \leqslant \delta$ thus establishing (5·3) *b*.

$$\text{Now } \frac{1}{2\pi B_n}\int_0^{\delta} \psi_{x_0}(t)\,\frac{|k_n(t)|}{\sin\frac{t}{2}}\,dt = \frac{1}{2\pi B_n}\int_0^{\frac{1}{n}} + \frac{1}{2\pi B_n}\int_{\frac{1}{n}}^{\delta}$$

$$= E_n + F_n \qquad (5\cdot4)$$

$$\text{By (5·3) } a,\ E_n \leqslant \frac{2n+1}{2\pi}\cdot\int_0^{\frac{1}{n}} \psi_{x_0}(t)\,dt.$$

and by Note 1, under B, hypothesis of $\bar{A}$ or A, namely (3·1), implies

$$n \int_0^{\frac{1}{n}} \psi_{x_0}(t)\, dt \to 0 \; n \to \infty.$$

Hence $$E_n \to 0 \text{ as } n \to \infty \tag{5·5}$$

$$F_n \leqslant \frac{k_3}{2\pi B_n} \cdot \int_{\frac{1}{n}}^{\delta} \psi_{x_0}(t) \cdot \frac{B_{1/t}}{t}\, dt \text{ by } 5·3\,(b)$$

$$\leqslant \frac{k_3}{2\pi B_n} \cdot \int_{\frac{1}{\delta}}^{n} \psi_{x_0}\left(\frac{1}{t}\right) \frac{B_t}{t}\, dt \text{ which by hypothesis of A}$$

converges to 0 with $n \to \infty$. (5·6)

Hence (5·2), (5·5) and (5·6) establish $\bar{A}$ (and with B, A also).

6. THEOREM C: If $f(x)$ be a periodic summable function and

(*a*) $\frac{1}{B_x} \int_0^x \theta(t) \cdot b_t\, dt \to 0$ or (*a'*) $\frac{1}{x} \int_0^x \theta(t)\, dt \to 0$ (*a''*) $\theta(x) \to 0$ as $x \to \infty$

where $$\theta(t) = \psi_{x_0}\left(\frac{1}{t}\right) \cdot \frac{B_t}{t \cdot b_t}, \tag{6·1}$$

and (*b*) the Fourier coefficients of $f(x) \leqslant \frac{A}{p_n}$, $(A > 0)$. (6·2)

then the Fourier series of $f(x)$ converges at x_0 to $f(x_0)$;

the terms b_t, B_t, p_n, $\psi_{x_0}(t)$ being as follows:—

[(1) $b_0 > 0$;

(2) $\frac{b_0}{b_1} \geqslant \frac{b_1}{b_2} \geqslant \cdots \geqslant \frac{b_n}{b_{n+1}} \to 1$ as $n \to \infty$;

(3) $\sum_0^n b_r = B_n \to \infty$ $n \to \infty$;

(4) $b_t = b_{[t]}$, $B_t = B_{[t]}$;

(5) p_n is defined by $\frac{B_{p_n}}{B_n} \leqslant \delta < \frac{B_{(p_n+1)}}{B_n}$, $0 < \delta < 1$, (δ being fixed);

(6) $\psi_{x_0}(t)$ as in 1·1]. (6·3)

Conditions (1), (2), (3) and (4) of (6·3) imply that b_n's and B_n's satisfy (1) and (2), (3) and (4) of 2.

Hence by theorems A and A_1, (6·1) implies that

$$\frac{1}{B_n}\sum_{0}^{n} b_r S_{n-r}(x_0) \to f(x_0) \text{ as } n \to \infty \qquad (6\cdot4)$$

In a paper[8] to be shortly published, I have established the following Tauberian Theorem:—"Under conditions (1) to (5) about b_n, if a sequence S_n is such that

$$(a)\ \frac{1}{B_n}\sum^{n} b_r S_{n-r} \to S \text{ as } n \to \infty \text{ and } (b)\ S_n - S_{n-1} \leqslant \frac{A}{p_n}\ (A>0)$$

then S_n converges to S in the ordinary sense. (6·5)

In view of this, conditions (6·2) and (6·4) establish theorem C.

7. *Special Cases of A and C.*—In these special cases we shall consider, the coefficients satisfy (6·3) thence the conditions of 2, so that we can consider the special cases of A and C, together.

(1) $b_n \sim \frac{1}{n^{\sigma}}$ $0 \leqslant \sigma < 1$. Theorem $\bar{A} \equiv$ (F. L. H.) theorem, and the Tauberian part of C = Cesaro-Tauber theorem for (C, r) $0 < r \leqslant 1$.

(2) $b_n = \frac{1}{n+1}$. Theorem A for this case is discussed in my paper[7] Theorem C in this case is the Hardy-Littlewood[5] (1932) test with slightly broader conditions.

(3) Corresponding to $b_n \sim \frac{1}{n \log n}, \frac{1}{n \log n \log\log n}$, etc., the point condition of A or A_1 gets stiffer and stiffer, and the Tauberian condition in C gets broader and broader, for example in the case $b_n \sim \frac{1}{n \log n}$, the point condition is in its simplest form is $\psi_{x_0}(t)$. $\log t \cdot \log|\log t| \to 0$ $t \to 0$.

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THE BRIGHTNESS OF THE ZENITH SKY DURING TWILIGHT—PART II

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Introduction

The measurements of the brightness of the zenith sky during evening twilight at Colaba, Bombay, during the clear season of 1937–38 were discussed by one of the authors in a previous paper (referred to hereafter as Part I).[1] For a comparison of the brightness of the twilight sky with that calculated from Rayleigh's theory of molecular scattering of light, it is of importance to carry out exact photometric measurements under conditions of very clear skies. The observations at Bombay were affected to an unknown extent by scattered light from dust or other particles suspended in the atmosphere due to stray illumination from the city lights. With a view to secure photometric measurements under conditions less affected by this, further experiments were carried out at two places: Poona (height above mean sea-level 1,830 ft.) and Mt. Sinhagad (height above mean sea-level 4,400 ft.). The present paper describes and discusses the results of these experiments carried out during the clear season of 1941–42.

Experimental

The photometric measurements were made with the same visual photometer discribed in Part I. The two sliding colour filters used in these experiments were: (1) Green, VG_1 (Schott and Genossen) and (2) Red, 27A (Wratten). Readings were taken for different positions of the sun below the horizon both before sunrise and after sunset during a large number of clear, moonless twilights. In these experiments the photometer lamp was fixed slightly nearer the magnesium oxide screen so that readings could be obtained for smaller depressions of the sun below the horizon (θ) than those obtained in Part I. Thus with the red filter, observations could be started when $\theta = 4°$ and continued down to $\theta = 11°$ while with the green filter they could be started when $\theta = 4° 30'$ and continued down to $\theta = 18°$ and even further. In taking measurements, one observer sat at the photometer in the dark room matching the brightness of the zenith sky with that of the Magnesium Oxide

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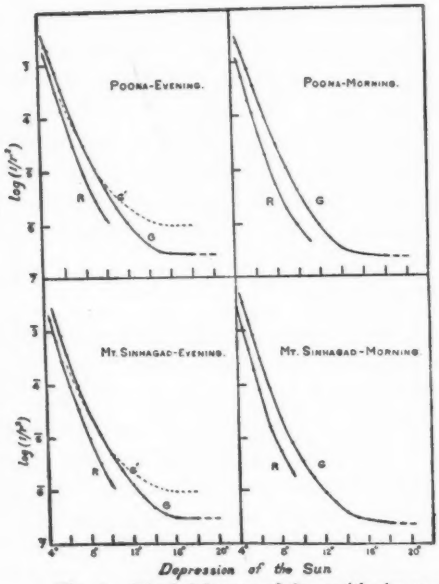
screen at short intervals of time, while the other observer sitting outside the room noted down the corresponding readings of the ammeter and the chronometer and also from time to time the condition of the sky.

It must be mentioned here that the black-out restrictions during the pariod of observation at Poona proved to be of great advantage as they eliminated almost completely the stray illumination of the lowest layers of the atmosphere. Conditions at Mt. Sinhagad were still better since there was no question of stray illumination at all and the place is above the dust layer.

The calibration of the photometer lamp was carried out as described before in Part I, against the same standard Kodak Acetylene lamp using the two colour filters. The factor K, referred to before, required to convert ammeter readings to units of brightness in candles per square centimeter per unit solid angle, is equal to 0·090 for the green filter VG_1 (Schott and Genossen) and 0·128 for the red filter 27A (Wratten).

Results



The readings were grouped separately according to time, place and colour filter used. For every twilight, two curves, one for each filter, were drawn showing the variation of log $(1/r^2)$ with the position (θ) of the sun below the horizon. The individual values in each group show little variation from the mean and therefore only an average curve was drawn for each group separately. Such average curves for Poona and Mt. Sinhagad are shown in Fig. 1. For the sake of comparison the Bombay evening curve for the green filter is also shown in the same figure.

Fig. 1. The brightness of the zenith sky as a function of the depression of the sun (in degrees) below the horizon

Brightness of zenith sky $= 0{\cdot}090/r^2$ for the green filter.

and " " $0{\cdot}128/r^2$ for the red filter.

To isolate the red region, Wratten red filter 27A and for the green region, Schott and Genossen green filter VG_1 were used. The broken curve represents observations made at Bombay with the same green filter.

The mean values of the intensities expressed in the above-mentioned units for different depressions of the sun below the horizon are given in Table I for each group separately. The numbers of observations from which the averages are obtained are given in brackets at the top of each column.

TABLE I

Logarithms of the brightness in candle power of the zenith sky at Poona and Mt. Sinhagad during twilight for different depressions of the sun below the horizon

Depression of the sun below the horizon	Poona				Mt. Sinhagad			
	Sunrise		Sunset		Sunrise		Sunset	
	Red n=(13)	Green (15)	Red (2)	Green (7)	Red (14)	Green (14)	Red (4)	Green (4)
4°	..	..	..	..	$\bar{4}\cdot538$	..	..	..
4° 30'	$\bar{4}\cdot141$	$\bar{4}\cdot322$	$\bar{4}\cdot288$	$\bar{4}\cdot302$	$\bar{4}\cdot148$	$\bar{4}\cdot462$	$\bar{4}\cdot358$	..
5°	$\bar{5}\cdot805$	$\bar{4}\cdot132$	$\bar{5}\cdot928$	$\bar{4}\cdot082$	$\bar{5}\cdot748$	$\bar{4}\cdot127$	$\bar{5}\cdot938$	$\bar{4}\cdot172$
5° 30'	$\bar{5}\cdot378$	$\bar{5}\cdot822$	$\bar{5}\cdot608$	$\bar{5}\cdot812$	$\bar{5}\cdot383$	$\bar{5}\cdot802$	$\bar{5}\cdot588$	$\bar{5}\cdot882$
6°	$\bar{5}\cdot148$	$\bar{5}\cdot522$	$\bar{5}\cdot238$	$\bar{5}\cdot502$	$\bar{5}\cdot008$	$\bar{5}\cdot432$	$\bar{5}\cdot258$	$\bar{5}\cdot532$
7°	$\bar{6}\cdot378$	$\bar{6}\cdot922$	$\bar{6}\cdot578$	$\bar{6}\cdot892$	$\bar{6}\cdot338$	$\bar{6}\cdot892$	$\bar{6}\cdot658$	$\bar{6}\cdot952$
8°	$\bar{7}\cdot828$	$\bar{6}\cdot422$	$\bar{6}\cdot018$	$\bar{6}\cdot432$	$\bar{7}\cdot868$	$\bar{6}\cdot322$	$\bar{6}\cdot108$	$\bar{6}\cdot432$
9°	$\bar{7}\cdot388$	$\bar{7}\cdot982$	$\bar{7}\cdot558$	$\bar{6}\cdot002$	$\bar{7}\cdot428$	$\bar{7}\cdot862$	$\bar{7}\cdot638$	$\bar{6}\cdot032$
10°	$\bar{7}\cdot098$	$\bar{7}\cdot582$	$\bar{7}\cdot248$	$\bar{7}\cdot632$	$\bar{7}\cdot088$	$\bar{7}\cdot487$	$\bar{7}\cdot248$	$\bar{7}\cdot652$
11°	$\bar{8}\cdot798$	$\bar{7}\cdot242$	..	$\bar{7}\cdot292$	$\bar{8}\cdot728$	$\bar{7}\cdot182$	..	$\bar{7}\cdot352$
12°	..	$\bar{8}\cdot926$	..	$\bar{7}\cdot012$	..	$\bar{8}\cdot912$	..	$\bar{7}\cdot092$
13°	..	$\bar{8}\cdot672$	..	$\bar{8}\cdot762$	..	$\bar{8}\cdot742$	..	$\bar{8}\cdot852$
14°	..	$\bar{8}\cdot492$	..	$\bar{8}\cdot572$	..	$\bar{8}\cdot532$	..	$\bar{8}\cdot632$
15°	..	$\bar{8}\cdot402$	..	$\bar{8}\cdot442$	..	$\bar{8}\cdot432$	..	$\bar{8}\cdot522$
16°	..	$\bar{8}\cdot372$	..	$\bar{8}\cdot402$	..	$\bar{8}\cdot402$	..	$\bar{8}\cdot462$
18°	..	$\bar{8}\cdot322$	..	($\bar{8}\cdot382$)	..	$\bar{8}\cdot332$	..	$\bar{8}\cdot452$

Discussion

For the sake of comparison, average values of intensities for $\theta = 9°$ to $\theta = 18°$ obtained at different stations are brought together in Table II, after converting them into star magnitudes using the standard candle as an

intermediate step. It will be seen from this table that in general, smaller intensities were observed at Poona and Mt. Sinhagad than at Bombay. Considering the green filter curves at Bombay the intial rapid fall of intensity continues down to about $\theta = 10°$ and then gradually reaches the final value of star magnitude 17·6 at about 15°–16°. But at Mt. Sinhagad and Poona the initial rapid fall of intensity continues down to about $\theta = 14°$ and then gradually attains the final value of about 19 star magnitude at $\theta = 16°$–$18°$. The measurements made by Link[2] at Rysy show that for $\theta = 13°$ the zenith sky there is brighter than that at Mt. Sinhagad and Poona for the same position of the sun below the horizon.

TABLE II

The average values of the intensities of the zenith sky expressed in star magnitude with respect to the standard candle, obtained at different stations for $\theta = 9°$ to 18°

Depression of the sun below the horizon	Poona				Mt. Sinhagad				Bombay		Rysy.
	Sunrise		Sunset		Sunrise		Sunset		Sunset		
	Red	Green	Red	Green	Red	Green	Red	Green	Red	Green	Green
9°	16·53	15·50	16·11	14·99	16·93	15·35	15·91	14·92	16·30	14·95	14·15
10°	17·25	16·05	16·88	15·92	17·28	16·28	16·88	15·87	..	15·74	15·00
11°	18·01	16·89	..	16·77	18·18	17·05	..	16·62	..	16·27	15·85
12°	..	17·69	..	17·72	..	17·72	..	17·24	..	16·72	16·55
13°	..	18·32	..	18·09	..	18·15	..	17·87	..	17·11	17·20
16°	..	19·07	..	18·89	..	18·95	..	18·85	..	17·60	..
18°	..	19·19	..	19·05	..	19·17	..	18·87	..	..	..

There is also a marked difference between the morning and the evening curves, the evening twilight being on the average slightly brighter than the morning twilight. This is probably due to the greater scattering of light by the dust in the atmosphere during evening twilight than during morning twilight: during day dust is raised from the ground layers by convection currents. In the evening it begins to settle down and this process continues throughout the night. As small suspended particles scatter the incident light enormously in the forward direction, it may be expected that the twilight glow is stronger when there is a greater amount of dust present in the upper atmosphere. The effect of the dust in reducing the intensity of the scattered light on its vertical path downwards from the zenith sky can be shown to be of much less importance. Hence the zenith sky during evening twilight should be brighter than during morning twilight. The larger

values of intensities observed at Bombay for $\theta = 16° - 18°$ arises from a different cause and are to be attributed to the scattering of the light from the city lights. As pointed out before the measurements at Poona and especially those at Mt. Sinhagad are less affected by this source of error.

The intensities observed at Poona and Mt. Sinhagad for $\theta = 18°$ (*viz.*, 18·87–19·19 star magnitude) though distinctly smaller are comparable with the value of 18·76 star magnitude observed by Jean Dufay[3] for the night sky during 1922–26, and with the mean value of 18·61* star magnitude obser-ed by Bauer, Danjon and Langevin[4], at Mont. Blanc for $\theta = 25°–29°$. Thus, Jean Dufay's values are about 1·28 times, and those of Bauer, Danjon and Langevin about 1·49 times the values obtained by us. It appears that both these workers had not used any colour filters in their experiments while we used a green filter VG_1 (Schott and Genossen) which transmits only wave-lengths between 4800 to 6000 Å.

To make a comparison of the absolute brightness of the night sky observed in India with those obtained in higher latitudes, it is proposed to continue these measurements of the twilight intensity further using the auroral filter.

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* This value was given as 18·47 star magnitude in the former paper owing to a mistake in calculation.

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